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THREE-DIMENSIONAL CHAINS AND THE ASSOCIATED COLLINEATIONS IN SPACE.

By Hazel Hope MacGregor.

Introduction.

Analytically the classification of the collineations in a linear space of any number of dimensions into types according to their invariant figures is equivalent to the classification of the linear homogeneous transformations in this space. The problem is thus intimately connected with the theory of elementary divisors of Weierstrass and on this analytic basis the classification for a space of n-dimensions has been made by Segré.*

In this work the coefficients of the transformation are supposed to be any complex numbers, and two collineations are regarded as equivalent if, and only if, one can be transformed into the other by a collineation with complex coefficients. From this point of view is obtained the well-known classification of nonsingular projective transformations on a line, in a plane, and in space; namely, two types on a line, five types in a plane, and thirteen types in space. This classification has also been made synthetically by Professor H. B. Newson.†

If, however, the coefficients in the collineation are restricted to be real numbers and if two such collineations are regarded as equivalent if, and only if, one can be transformed into the other by a transformation with real coefficients, the number of types is increased. Thus in the one-dimensional case we have the three well-known types usually designated by the terms hyperbolic, elliptic, and parabolic. Here, furthermore, arises the important additional problem as to the conditions under which a collineation with complex coefficients can be transformed into one with real coefficients. In the one-dimensional case again, these conditions are well known, it being necessary and sufficient that the projective transformation on the complex variable leave a circle in the Argand plane of the complex variable invariant.

In a recent paper‡ Professor J. W. Young has considered these problems for the one-dimensional case from the point of view of Projective Geometry,

* C. Segre, Memorie della R. Accademia delle Scienze di Torino. 1885.

† H. B. Newson, "Types of Projective Transformation in the Plane and in Space." Kansas University Quarterly, vol. VI, pages 64–69.

‡ J. W. Young, "The Geometry of Chains on a Complex Line." *Annals of Mathematics*, Second Series, vol. 11, October, 1909, pp. 33–48. This paper will be referred to as (A).

the notion of a linear chain being fundamental. In a subsequent paper* by making use of the idea of the two-dimensional chain Professor Young considers these problems for the collineations of the plane. He finds in all six different types of collineations in a plane which leave two-dimensional chains invariant and which may therefore be represented with real coefficients.

It is my purpose in this paper to apply this principle of classification to the nonsingular collineations in a complex space of three dimensions. To do this it will be necessary to introduce the *spatial*, or *three-dimensional* chain; the latter may be described as any class of points, lines, and planes in space which may be obtained from the class of real points, lines, and planes by a projective collineation. After defining a three-chain synthetically and observing some of its fundamental properties (§ 1) I give (§ 2) the classification of the collineations in space which leave three-dimensional chains invariant and derive the necessary and sufficient conditions that a collineation be of this type. Any such collineation may be represented with real coefficients.

§ 1. Definitions and Fundamental Properties of Three-Dimensional Chains.

The definitions and fundamental properties of linear and planar chains are assumed known. All points, lines, and planes considered are in the same complex three-dimensional space.

Definitions.—A point is said to be *linearly related* to two planar chains $\mathfrak{C}_1^2$ and $\mathfrak{C}_2^2$ on distinct planes and having a linear chain $\mathfrak{C}$ in common, provided it is the intersection of two lines each of which joins a point of $\mathfrak{C}_1^2$ to a distinct point of $\mathfrak{C}_2^2$. A line (plane) is said to be *linearly related* to $\mathfrak{C}_1^2$ and $\mathfrak{C}_2^2$ if it contains two distinct (three non-collinear) points which are linearly related to $\mathfrak{C}_1^2$, $\mathfrak{C}_2^2$. The set of all points, lines, and planes linearly related to $\mathfrak{C}_1^2$ and $\mathfrak{C}_2^2$ is called the *three-dimensional chain*, or more briefly, the *three-chain* defined by $\mathfrak{C}_1^2$ and $\mathfrak{C}_2^2$. This definition is equivalent to the analytic definition suggested in the introduction.

Fundamental Properties of Three-Chains.

I. *Through any five points, no four of which are coplanar, passes one and only one three-chain.*

This follows at once from the fact that by the Fundamental Theorem of Projectivity a projective collineation in space is uniquely determined

* J. W. Young, "Two-Dimensional Chains and the Associated Collineations in a Complex Plane." Transactions of the American Mathematical Society, vol. XI, 1910, pp. 280–293. Referred to as (B).

by five pairs, AA', BB', etc., of homologous points, no four of either of the sets AB . . . or $A'B'$ . . . being coplanar.

II. *Any class of points, lines, and planes, in space projective with a three-chain is a three-chain.*

The next five properties are among what may be termed the *internal properties* of a three-chain. They follow at once from the well-known properties of the real three-space.

III. *Any two coplanar lines of a three-chain meet in a point of the three-chain.*

IV. *Any line of a three-chain meets any plane (not containing the line) of the three-chain in a point of the three-chain.*

V. *Any two distinct planes of a three-chain meet in a line of the three-chain.*

VI. *Any line of a three-chain has the points of a linear chain in common with the three-chain, and only these.*

VII. *A plane of a three-chain has the points and lines of a planar chain in common with the three-chain, and only these.*

The following properties may be termed *external properties* of a three-chain.

VIII. *Any three-chain is met by a plane not of the three-chain in a linear chain.*

Proof.—Let $ax_1 + bx_2 + cx_3 + dx_4 = 0$ be any plane in the complex space, where $a = a' + ia''$, $b = b' + ib''$, and so on, the a', a'', b', b'', . . . being real numbers. Then if x_1, x_2, x_3, x_4, are the coördinates of a point of the real space we have

$$a'x_1 + b'x_2 + c'x_3 + d'x_4 = 0,$$

$$a''x_1 + b''x_2 + c''x_3 + d''x_4 = 0.$$

These two equations have real coefficients and represent real planes; their intersection is a real line. But $ax_1 + bx_2 + cx_3 + dx_4 = 0$ is the equation of a plane passing through the intersection of these two planes. Hence every plane contains a real line and so by projection every plane has a linear chain in common with any three-chain.

Definition.—Two points are said to be *conjugate* to each other with respect to a given three-chain if they are the points into which a pair of conjugate imaginary points are transformed when the real three-chain is transformed into the given three-chain. Any point of a three-chain is *conjugate* with itself. The given three-chain is said to be *about* the points. Two lines are said to be *conjugate* with respect to a given three-chain if they are so related that every point on the one is the conjugate of some point on the other, in which case the three-chain is said to be *about* the two lines. Two projectivities on conjugate lines are said to be *conjugate* with

respect to the three-chain if they are so related that any two conjugate points are homologous with two conjugate points.

IX. *Any line joining two points conjugate to each other with respect to a given three-chain is a line of the three-chain.*

It is sufficient to recall that any line joining two conjugate imaginary points of real space is a real line, as the theorem then follows at once.

X. *A line joining any two points is conjugate with respect to a three-chain to a line joining the points conjugate to the two given points.*

XI. *Through any point not in a given three-chain there is one and only one line of the three-chain.*

If there were more than one, the point would be a point of the three-chain, and there is one, since the line joining a point to its conjugate point with respect to the three-chain is a line of the three-chain.

§ 2. A Classification of the Collineations in Space with Reference to their Invariant Three-Chains.

In this classification of the collineations in space with reference to their invariant three-chains, I shall show first, under what geometric conditions each type of collineation, based on a classification according to the invariant figure, will leave a three-chain invariant,—thus subdividing the type; then I shall show what restrictions must be placed on the three-chain in order that it remain invariant under this subtype.

Two theorems of frequent application are the following:

If a point P of an invariant three-chain of a collineation is not on an invariant line, any invariant plane through P is a plane of the three-chain.

Let P be transformed by the collineation into P' and P' into P''. The points P, P', P'' are not collinear but are the points of an invariant three-chain, and in an invariant plane, and so determine it as a plane of the three-chain.

If an invariant three-chain contains an invariant point, it contains an invariant planar chain, since point and planar chain are dual notions.

In the following, the results and terminology of papers (A) and (B) cited in the introduction are assumed known.

Type [1, 1, 1, 1].—The invariant figure of Type [1, 1, 1, 1] consists of the vertices, edges, and faces of a tetrahedron. The two-dimensional transformation in each invariant plane leaves a triangle invariant. The one-dimensional transformation on each invariant line leaves two elements invariant. Analytically this collineation is characterized by the fact that its characteristic equation has four distinct roots. If we assume the invariant three-chain to be the real three-chain, the coefficients of the characteristic equation are all real numbers and we have three cases to consider

according as the roots of the characteristic equation are all real, two real and two imaginary, or all imaginary. So any invariant three-chain contains four, two, or no double points of the collineation.

Type [1, 1, 1, 1]h.—Suppose the invariant three-chain to contain the four double points A, B, C, D. Then the projectivities on the *six* invariant lines are all hyperbolic. For the invariant line AB containing two points of the three-chain contains an invariant linear chain of the three-chain through A and B (A, p. 42) and similarly for BC, CA, AD, BD, and CD. The invariant three-chain contains an invariant planar chain on each of the invariant planes, since it contains three invariant points on each such plane.

The necessary and sufficient condition that a collineation of Type [1, 1, 1, 1] leave invariant a three-chain containing A, B, C, and D, is that the two-dimensional transformations in two of the invariant planes be hyperbolic of Type [1, 1, 1] (B).

That it is sufficient follows from the fact that two invariant planar chains determine the three-chain as invariant.

Through any point in space, not in an invariant plane, there is under this type of collineation one and only one invariant three-chain (I)* (cf. Theorem II (B)).

Type [1, 1, 1, 1]h-e.—If the invariant three-chain contains only two of the double points of the collineation, B and C, then it contains an invariant linear chain containing B and C, so the projectivity on BC is hyperbolic (A). The points A and D are then conjugate with respect to the invariant three-chain, and the line AD joining them must be a line of the three-chain (IX), so that AD contains an invariant linear chain of the invariant three-chain which does not contain the double points A and D. Therefore the projectivity on AD is elliptic (A).

The invariant three-chain contains invariant planar chains on BAD and on CAD, since each contains a point and line of the invariant three-chain.

The necessary and sufficient condition that a collineation of Type [1, 1, 1, 1] leave invariant a three-chain containing B and C, but not containing A and D, is that the two-dimensional transformations in ADB and ADC be elliptic of Type [1, 1, 1] (B).

It is sufficient since two invariant planar chains determine an invariant three-chain.

Through any point P in space not in an invariant plane there is under a collineation of this type one, and only one, invariant three-chain containing B and C, and about A and D.

Let BP meet ADC in a point B' and let CB' meet AD in B''. C, B', B'' determine a unique planar chain about A, D, which is invariant. A similar

* Roman numerals in ( ) refer to the fundamental properties in § 1.

invariant planar chain is determined in BAD. These two planar chains determine uniquely the invariant three-chain.

Type [1, 1, 1, 1]e.—Here the invariant three-chain contains none of the double points of the collineation. A, D and B, C are two pairs of points conjugate with respect to the invariant three-chain and the projectivities on AD and BC are elliptic. But this condition alone is not sufficient to characterize the collineation. To do this it is necessary to consider the projectivities on a pair of conjugates lines, such as AB and CD. Let P be any point of the invariant three-chain not on an invariant plane. Pass a plane through P and CD meeting AB in X, and a plane through P and AB meeting CD in Y. These two planes, being conjugate with respect to the three-chain, meet in a line of the three-chain, so X and Y are conjugate with respect to the three-chain. Now if P is transformed into P' and X into X', then Y is transformed into Y', the conjugate of X'.

The necessary and sufficient condition that a collineation of Type [1, 1, 1, 1] leave invariant a three-chain $\mathfrak{C}^3$ about the points A, D and about the points B, C is that the projectivities on AD and BC be elliptic and the projectivities on AB and CD be conjugate with respect to $\mathfrak{C}^3$.

That this is a sufficient condition may be seen as follows. Let P be any point not on an invariant plane. Draw the line through P meeting BC and AD. These points of intersection will determine linear chains about B, C and about A, D and the two linear chains and P determine a three-chain. This three-chain $\mathfrak{C}^3$ is invariant under the collineation, for P is on a line joining a point S of $\mathfrak{C}^3$ on AD to a point T of $\mathfrak{C}^3$ on BC and also on a line joining a point X of AB to the conjugate of X with respect to $\mathfrak{C}^3$ on CD. The collineation will put S and T into the points S' and T' of $\mathfrak{C}^3$, and it will put X into X' and Y into Y', the conjugate of X' with respect to $\mathfrak{C}^3$. $X'Y'$ and $S'T'$ are lines of $\mathfrak{C}^3$, so their intersection is a point of $\mathfrak{C}^3$, and therefore $\mathfrak{C}^3$ is invariant.

Through any point of space not on an invariant plane there is one and only one invariant three-chain of this type.

Let O be any point besides P, not on an invariant plane, and let π_1 be the projectivity leaving A, B, C, D invariant and putting P into O, and let π be the original collineation. Now $\pi_1\pi\pi_1^{-1} = \pi$, since the two projectivities have the same invariant points. But $\pi_1\pi\pi_1^{-1}$ leaves the three-chain through O invariant; therefore π leaves it invariant. That there is only one such three-chain follows readily from the proof of the last theorem.

If a collineation of Type [1, 1, 1, 1] leaves a three-chain invariant which contains the four invariant points A, B, C, D, the collineation may conveniently be designated as *hyperbolic of Type [1, 1, 1, 1]*, or *Type [1, 1, 1, 1]h*; if it contains only two of the invariant points, as *hyperbolic-elliptic of Type*

[1, 1, 1, 1], or *Type [1, 1, 1, 1]h-e*; and if it contains no invariant points as *elliptic of Type [1, 1, 1, 1]*, or *Type [1, 1, 1, 1]e*.*

Type [2, 1, 1].—The invariant figure defining a collineation of Type [2, 1, 1] consists of three noncollinear points A, B, C; four lines, the sides of the triangle ABC, and a line l, through A and not in the plane ABC; and three invariant planes determined by l, AB, and AC.

If the invariant three-chain is the three-chain of real points it must contain either one or three of the invariant points of the collineation. The characteristic equation has real coefficients and so at least one real root. By the principle of duality it must contain either one or three of the invariant planes through A.

Type [2, 1, 1]h.—If an invariant three-chain contains the three double points of the collineation the projectivities on AB, AC, and BC are hyperbolic and on l parabolic. Conversely, if the projectivities on AB, AC, and BC are hyperbolic any three-chain containing A, B, C, and an invariant linear chain on l is invariant, being determined by an invariant planar chain on ABC and an invariant linear chain on l.

Through any point P, not in an invariant plane, there passes one and only one three-chain invariant under a collineation of this type.

Pass a plane through P and BC. It cuts l in a point, thus determining an invariant linear chain on l. Pass a plane through P and l. It cuts BC in a point which determines an invariant linear chain on BC. These two linear chains with P determine a three-chain $\mathfrak{C}^3$ which is invariant. To prove this draw a line through P and a point of the linear chain on l. It will cut the plane ABC in a point of $\mathfrak{C}^3$, which with A and the linear chain on BC determines an invariant planar chain on ABC. The invariant linear chain on l and the invariant planar chain on ABC determine the invariant three-chain.

The necessary and sufficient condition that a collineation leave invariant a three-chain containing A, B, C is that the two-dimensional projectivity in the plane ABC be hyperbolic of Type [1, 1, 1].

A collineation satisfying the conditions of this theorem may be designated as hyperbolic of Type *[2, 1, 1]*.

The necessary and sufficient condition that a three-chain be invariant under a collineation of Type [2, 1, 1]h is that it contain A, B, C and an invariant linear chain on l.

Type [2, 1, 1]e.—The invariant three-chain contains only one of the double points of the collineation, namely A, corresponding to the double root of the characteristic equation.

* It should perhaps be noted that these types are not mutually exclusive. They overlap when one or more of the projectivities on invariant lines are involutoric.

Any invariant three-chain containing A, but not B and C, contains an invariant planar chain on the plane ABC and an invariant linear chain on l, each containing A.

Since B and C are conjugate with respect to the invariant three-chain $\mathfrak{C}^3$, BC is a line of $\mathfrak{C}^3$ and this line with A determines an invariant planar chain of $\mathfrak{C}^3$ in the plane ABC. So the projectivity in ABC is elliptic of Type [1, 1, 1] (B). $\mathfrak{C}^3$ also has an invariant linear chain on l, for the plane ABl has at least a linear chain in common with any three-chain (VIII). Assume that it has a linear chain in common with $\mathfrak{C}^3$, not on l or AB (the latter is clearly impossible), but on some other line l'. This line is transformed into another line l'' which will also have a linear chain in common with $\mathfrak{C}^3$ and ABl. Both $\mathfrak{C}^3$ and the plane ABl are invariant under the collineation and these two linear chains determine a planar chain on ABl and therefore $\mathfrak{C}^3$ contains an invariant planar chain on ABl, which is impossible. Therefore $\mathfrak{C}^3$ must contain a linear chain on l, and since the projectivity on l is parabolic, this chain contains A. Therefore

The necessary and sufficient condition that a collineation of Type [2, 1, 1] leave invariant a three-chain containing only one of the double points is that the collineation in the plane ABC be elliptic of Type [1, 1, 1].

We will designate this collineation as *elliptic of Type* [2, 1, 1] or *Type* [2, 1, 1]*e*.

Through any point P in space, not in an invariant plane, passes one and only one invariant three-chain of this type.

The proof is similar to that for the corresponding theorem under Type [2, 1, 1]*h*.

The necessary and sufficient condition that a three-chain be invariant under a collineation of Type [2, 1, 1]e is that it contain a planar chain containing A and about B and C, and an invariant linear chain on l.

Type [2, 2].—The invariant figure of this collineation consists of two points, A, B, two planes, ABl and ABm, and three non-coplanar lines, AB, l, m, of which l contains A and m contains B. The two-dimensional transformations in ABl and ABm are both of the second type (B). The one-dimensional transformations are parabolic on l and m and of the first type on AB (A). The characteristic equation has two double roots, so that we have two cases to consider. Either the invariant three-chain contains the two double points, A and B, or else A and B are conjugate with respect to the invariant three-chain.

Type [2, 2]**h**.—If the invariant three-chain contains A and B, it contains two invariant planar chains.

The necessary and sufficient condition that a collineation of Type [2, 2] leave a three-chain through A and B invariant is that the projectivity on AB be hyperbolic.

A collineation satisfying this condition we will call *hyperbolic of Type* [2, 2] or *Type* [2, 2]*h*.

The necessary and sufficient condition that a three-chain be invariant under a collineation of Type [2, 2]*h is that it contain an invariant linear chain on* l *and* m.

If a three-chain contains these two linear chains it has an invariant planar chain on both ABl and ABm. These two invariant planar chains intersect in an invariant linear chain on AB and determine an invariant three-chain. That it is a necessary condition follows from the fact that an invariant three-chain containing A and B must contain two invariant planar chains.

Through any point P *in space, not in one of the invariant planes, there passes one and only one of these invariant three-chains.*

Let P' be the point into which P is transformed. Then $PP'A$ cuts m in a point of the three-chain determining the invariant linear chain on m, and $PP'B$ cuts l in a point determining the invariant chain on l. If we pass a plane through PP' and a point of the chain on l, it will cut AB in a point of the three-chain and so determine the linear chain on AB.

Type [2, 2]e.—If the invariant three-chain does not contain A and B, it contains no invariant planar chain and no invariant linear chains on l and m.

The necessary and sufficient condition that a collineation of Type [2, 2] *leave invariant a three-chain* $\mathfrak{C}^3$ *not containing the double points,* A *and* B, *is that the projectivity on* AB *be elliptic and the projectivities on the two double lines* l *and* m *be conjugate with respect to* $\mathfrak{C}^3$.

A collineation satisfying the above conditions we will call *elliptic of Type* [2, 2] or *Type* [2, 2]*e*.

There is one and only one invariant three-chain through every point not on a double plane.

These theorems may be proved as the similar theorems concerning the elliptic collineations of Type [1, 1, 1, 1].

Type [3, 1].—In this collineation we have two invariant points A and B, two invariant lines AB and l, passing through A, and two invariant planes, β determined by AB and l and α containing l. The two-dimensional projectivity in the plane β is of the second type and that in α of the third type. Three of the roots of the characteristic equation coincide so that all the roots are real. Any invariant three-chain therefore contains A and B, and by the principal of duality a planar chain on α, and one on β.

The necessary and sufficient condition that there be an invariant three-chain under a collineation of Type [3, 1] *is that the projectivity on* AB *be hyperbolic.*

Through any point P, *not in* α *or* β, *there is under this collineation one and only one invariant three-chain.*

Type [4].—The collineation of this type has as its invariant figure a single invariant point A in a single invariant plane α, and a single invariant line l in the invariant plane and through the invariant point. The two-dimensional transformation in the invariant plane is of the third type, and the one-dimensional transformation along the invariant line is parabolic. Any invariant three-chain contains A, since the roots of the characteristic equation all coincide, and, by the principle of duality, an invariant planar chain on α.

Every collineation of Type [4] leaves invariant every three-chain containing an invariant planar chain on α, and a pair of homologous points not in α.

Let P and P' be a pair of homologous points not on α; PP' cuts α in a point Q not on l. Let Q be transformed into Q'. QQ' cuts l in a point S, and S is transformed into S'. Let $\mathfrak{C}^3$ be the three-chain defined by the planar chain $|QQ'S'A|$ and PP'. $\mathfrak{C}^3$ is invariant. For $|QQ'S'A|$ is invariant (B). Now $PP' = PQ$ is transformed into $P'Q'$. The linear chain of planes on l in $\mathfrak{C}^3$ is invariant, since it contains α and a pair of homologous planes. Hence the plane lP' is transformed into lP'', a plane of $\mathfrak{C}^3$. The intersection of P'' and $P'Q'$ is the point P'' into which P' is transformed and is a point of $\mathfrak{C}^3$. Therefore $\mathfrak{C}^3$ is invariant and it is the only invariant three-chain through P, since every such three-chain must contain P', Q, Q' and therefore the planar chain on α. This is sufficient to determine the three-chain.

Perspective Collineations in Space.

A perspective collineation is determined by a plane of invariant points α, a center A which is invariant, and a pair of homologous points. There are two types of perspective collineations, one called "the homology in space," where A is not in α, and the other called "the elation in space," where A is in α.

Type [(1, 1, 1) 1].—In the case of the homology the one-dimensional transformations on the invariant lines through A and in the invariant pencils of rays and planes whose vertices and axes are in α are all of the first type, leaving two elements invariant.

Every invariant three-chain must contain A and a planar chain on α.

Let $\mathfrak{C}^3$ be any invariant three-chain. Let RR', PP', QQ' be any three pairs of homologous points of $\mathfrak{C}^3$ which are not all coplanar. Every pair of homologous points is collinear with A; therefore A is a point of $\mathfrak{C}^3$ (III). The dual argument proves that $\mathfrak{C}^3$ contains a planar chain on α.

A homology leaves invariant a three-chain if and only if the projectivity on a line through the center is hyperbolic.

Let $\mathfrak{C}^2$ be any planar chain on α and B be a point of $\mathfrak{C}^2$. Let P be any

point of the line BA, except B and A, and let the three-chain defined by $\mathfrak{C}^2$, A and P be denoted by $\mathfrak{C}^3$. Then $\mathfrak{C}^3$ is invariant, if and only if the projectivity on AP is hyperbolic. For then $|APB|$ would be invariant and P', the point into which P is transformed, would be in $\mathfrak{C}^3$. Since $\mathfrak{C}^3$ is determined by $\mathfrak{C}^2$, A and P, this proves also that $\mathfrak{C}^3$ is invariant.

Type [(2, 1, 1)].—In the case of the elations the one-dimensional transformations along all invariant lines through A and in all the invariant pencils of lines and planes whose vertices and axes are in α are parabolic.

By methods similar to those used under Type [(1, 1, 1), 1] it is easily proved that

The necessary and sufficient condition that a three-chain be invariant under a space elation is that it contain A, a planar chain on α, and an invariant linear chain on a line through A not in α. Every collineation of this type without restriction leaves a three-chain invariant.

The results for the remaining types can readily be obtained by the methods used hitherto. It seems desirable, therefore, to omit all proofs in the sequel.

Type [(1, 1), 1, 1].—The invariant figure of this type consists of all the points on a line AD, and two points, B and C, such that A, B, C, D, are not coplanar. The two-dimensional projectivities in the planes BAD and CAD are homologies, while those in the other invariant planes are of the first type.

Types [(1, 1), 1, 1]**h and** [(1, 1), 1, 1]**e**.—*If an invariant three-chain $\mathfrak{C}^3$ contains B and C, the two-dimensional projectivity in any plane through BC and a point of $\mathfrak{C}^3$ not on BC must be hyperbolic of Type* [1, 1, 1]; *if $\mathfrak{C}^3$ does not contain B and C, the two-dimensional projectivity in BPC must be elliptic of Type* [1, 1, 1].

If the invariant three-chain contains B and C the collineation may be called *hyperbolic of Type* [(1, 1), 1, 1]; if it does not contain these points it may be called *elliptic of Type* [(1, 1), 1, 1].

Any three-chain containing a linear chain through B and C, or about B and C, and a linear chain on the line AD is invariant.

Type [2, (1, 1)].—The invariant figure of this collineation consists of all the points on a line BC and a point A, not on BC, and a line l through A not meeting BC. The two-dimensional transformation in the plane ABC is a homology and that in any invariant planes containing l is of Type [2, 1].

The necessary and sufficient condition that a collineation of Type [2, (1, 1)] *leave a three-chain invariant is that the collineation in the plane ABC be hyperbolic. The necessary and sufficient condition that a three-chain be invariant under this collineation is that it contain a planar chain on ABC, containing A and the line BC, and an invariant linear chain on l.*

Type [(1, 1), (1, 1)].—This type is determined by an invariant figure consisting of all the points on two nonintersecting lines, l and m, and of all lines and planes thereby determined. The one-dimensional transformations along the invariant lines are all of the first type, and the two-dimensional transformations in all the invariant planes are homologies. It can easily be proved that the projectivity on any invariant line which is not pointwise invarient, is projective with the projectivity on any other such line. So if the projectivity on one such invariant line is hyperbolic (elliptic) then the projectivity on every other such line is hyperbolic (elliptic).

The necessary and sufficient condition that a collineation of Type [(1, 1), (1, 1] *leave invariant a three-chain containing an invariant point is that the projectivity on an invariant line, not l or m, be hyperbolic, and that it leave invariant a three-chain not containing an invariant point, is that the projectivity on an invariant line, not l or m, be elliptic.*

Such a collineation we will call *hyperbolic of Type* [(1, 1), (1, 1)] if it contains an invariant point, otherwise we will call it *elliptic of Type* [(1, 1), (1, 1)].

The necessary and sufficient condition that a three-chain be invariant under the hyperbolic (elliptic) collineation of Type [(1, 1), (1, 1)] *is that it contain a linear chain through (about) the double points on each of two invariant lines, not l or m.*

Type [(2, 1), 1].—In this type all the points on a line AC, and a point B, not on AC, are invariant, and all the lines through A and lying in a plane π, containing AC but not B, are invariant. The two-dimensional transformation in the plane α, determined by B and AC, is a homology; that in π is an elation, and that in every plane through AB, except α, is of the second type.

A collineation of this type leaves a three-chain invariant, provided only the homology on α is hyperbolic and the necessary and sufficient condition on the three-chain that it remain invariant is that it contain A, B, and an invariant planar chain on π.

Type [(2, 2)].—In this type all the points on a line AC and all the planes of the pencil having this line as an axis are invariant. In each invariant plane all the lines through some point of the axis will be invariant and no two of these centers coincide. The two-dimensional projectivities in the invariant planes are all elations.

This type, without any restriction, leaves a three-chain invariant and the necessary and sufficient condition that a three-chain remain invariant is that it contain an invariant linear chain on each of two non-coplanar invariant lines.

Type [(3, 1)].—In this type the invariant figure consists of all the points

on a line AB, all the planes on an axis l through A, and all the lines through A in the plane β, determined by AB and l. The two-dimensional transformation in the plane β is an elation and that in all the other invariant planes is of the third type.

There is no restriction on the collineation in order that it leave a three-chain invariant. The necessary and sufficient condition that a three-chain be invariant is that it contain an invariant linear chain on l.

We have thus seen that the collineations in space that leave a three-chain invariant may be classified into nineteen subtypes under the thirteen general types. These nineteen subtypes are distributed as follows: three of Type [1, 1, 1, 1]; two of each of the Types [2, 1, 1], [2, 2] [(1, 1), 1, 1], and [(1, 1), (1, 1)]; and one of each of the other eight Types.

University of Kansas,
June, 1911.

DETERMINATION OF THE CONSTANTS IN EULER'S PROBLEM CONCERNING THE MINIMUM AREA BETWEEN A CURVE AND ITS EVOLUTE.*

By E. J. Miles.

The evolute problem in the calculus of variations is stated as follows: Given two points P_0 and P_1 and directed lines P_0Q and QP_1 through them, to determine an arc tangent to these two lines at P_0 and P_1, which with its evolute and its normals at P_0 and P_1 will enclose the minimum area.

The problem was first given by Euler in his Methodus Inveniendi, etc., p. 64, 1744. It has also been discussed by several later writers, among whom may be mentioned Lindelöf and Moigno† and Kneser.‡

The only curves which can give a minimum are known to be cycloids. The question then naturally arises: *Is it possible to join any two points a and b by a cycloid which is tangent to two given directed lines through a and b?* This is the question which will here be discussed.

For convenience the equation of the cycloid will be taken in the form

$$C: \begin{cases} x = t - \sin t = \pi + 2u + \sin 2u, \\ y = -1 + \cos t = -(1 + \cos 2u), \end{cases}$$

where u is the angle through which the tangent line has turned, starting from $-\pi/2$ (see figure 1). The first arch will then correspond to the interval

$$-\frac{\pi}{2} \leqq u \leqq \frac{\pi}{2}.$$

The following problem is then proposed: Is it possible to find two points A and B on this cycloid such that the tangents at these points intersect at the same angle as the two given directed lines through a and b? If this problem can be solved, then the triangle ABC can be transformed by a similarity transformation into a triangle of the same size as abc and then by a movement of the plane it can be made to coincide with abc itself. Both of these transformations transform cycloids into cycloids and the original cycloid C will therefore go into one which satisfies the conditions of the original problem.

* Reported to the Chicago section of the Society, January 1, 1909.

† Calcul des Variations, p. 246, 1861.

‡ Lehrbuch der Variationsrechnung, pp. 203, 219, 1900.

In order to find the angles α and β for which there exists a triangle ABC on the cycloid C, one may derive the values of the angles α, β in terms of the parameter values u, v of the points A and B, and then study the transformation of the uv-plane into the $\alpha\beta$-plane. Different cases arise according as the arc AB contains a cusp or not. In § 1 the arc AB is supposed to con-

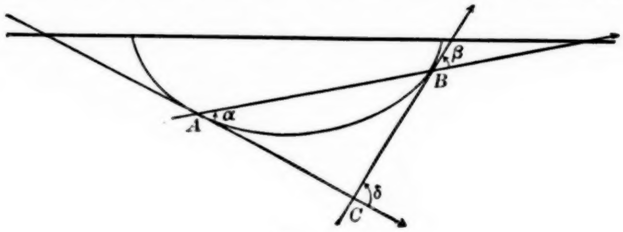

Fig. 1.

tain no cusp, and for convenience it will be taken on the first arch of the cycloid. This is the only case admitted by the variation problem for it has been shown that if the arc contains a cusp it does not furnish a minimum. In § 2 AB is supposed to contain n cusps and the discussion is divided into two cases according as δ is positive or negative.

§ 1. Solution of the problem when the arc AB contains no cusps.

Let u and v be the parameter values of A and B and therefore in this case the angles which the positive tangents at A and B make with the positive x-axis, φ the angle which the line AB makes with the x-axis and α, β, δ the angles indicated in Fig. 1.

If u and v are the parameter values of $A(x_1, y_1)$ and $B(x_2, y_2)$ respectively then we have

$$(1) \qquad \tan\varphi = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\sin(v-u)\sin(v+u)}{v - u + \sin(v-u)\cos(v+u)}.$$

From the figure it readily follows that the transformation of the uv-plane into the $\alpha\beta$-plane has the form

$$(2) \qquad \begin{cases} \alpha = \varphi - u, \\ \beta = v - \varphi. \end{cases}$$

The uv region which it is proposed to transform is T_1 of Fig. 2, defined by the inequalities

$$(3) \qquad -\frac{\pi}{2} \leqq u < v, \quad -\frac{\pi}{2} < v \leqq \frac{\pi}{2},$$

and corresponding to points A and B on the first arch of the cycloid.

Consider the line MN

(4) $$v - u = \delta,$$

where δ is a constant. It is transformed into the line $M'N'$

(5) $$\alpha + \beta = \delta,$$

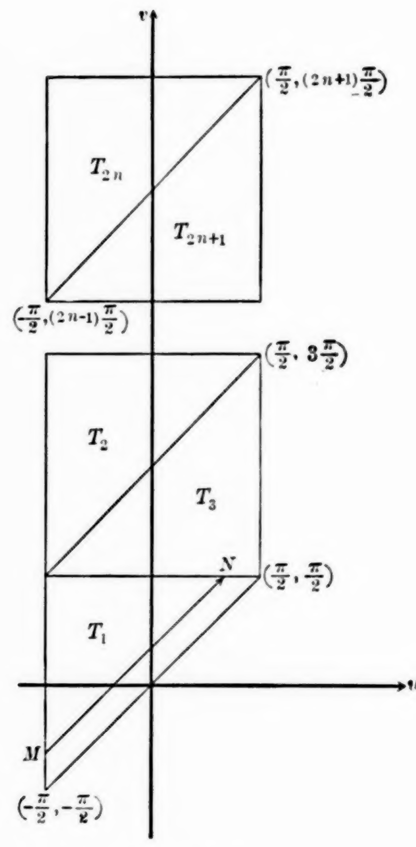

FIG. 2.

(see Fig. 3). The values of α corresponding to points on this line are defined by the equation

(6) $$\tan \alpha = \tan (\varphi - u) = \frac{-\delta \sin u + \sin \delta \sin (u + \delta)}{\delta \cos u + \sin \delta \cos (u + \delta)} \equiv F(u).$$

On the line (4) u has the limits

$$-\frac{\pi}{2} \leqq u \leqq \frac{\pi}{2} - \delta.$$

For these values $F(u)$ decreases monotonically from a positive maximum value α_1 to a positive minimum value α_2. For of the three expressions

$$F\left(-\frac{\pi}{2}\right) = \frac{\delta - \sin\delta\cos\delta}{\sin^2\delta} = \tan\alpha_1, \tag{7}$$

$$F\left(\frac{\pi}{2} - \delta\right) = \frac{\sin\delta - \delta\cos\delta}{\delta\sin\delta} = \tan\alpha_2, \tag{8}$$

$$F'(u) = \frac{\sin^2\delta - \delta^2}{[\delta\cos u + \sin\delta\cos(u + \delta)]^2},$$

the first two are positive and the third negative for δ between its limiting values 0 and π (see Fig. 1).

It is readily seen that the numerators are positive or negative by differentiating each until a derivative of constant sign is found, and considering the initial values. For example suppose we consider the numerator of $F'(u)$,

$$N(\delta) = \sin^2\delta - \delta^2, \quad N'(\delta) = 2(\sin\delta\cos\delta - \delta),$$
$$N''(\delta) = -2\sin^2\delta.$$

The second derivative is always negative or zero and so it follows that the function N itself has a maximum equal to zero for $\delta = 0$. The quotient $F(u)$ cannot become infinite for $0 < \delta < \pi$, $-\pi/2 \overline{\gtrless} u \overline{\gtrless} \pi/2 - \delta$, for both u and $u + \delta$ are then between $-\pi/2$ and $+\pi/2$. Consequently both terms in the denominator of $F'(u)$ are always positive and they cannot vanish simultaneously.

By a comparison of the equations (7) and (8) it is easily verified that

$$\tan\alpha_2 = \tan(\delta - \alpha_1). \tag{9}$$

This shows that the figure is symmetric with respect to the bisector of the first quadrant of the $\alpha\beta$-plane. So having computed α_1 the range of values of α follows. Then from the equation (5) one finds the end values of β. By means of the formulæ (7), (8), (9) the table given below is readily computed.

Table A.

δ	$\alpha_1 = \beta_2$	$\alpha_2 = \beta_1$
0.	0.	0.
.524	.348	.176
.755	.472	.283
1.047	.686	.361
1.571	1.004	.567
2.094	1.280	.814
2.356	1.397	.959
2.618	1.489	1.129
3.142	1.571	1.571

In this table the angles are expressed in radians. From these values the image I in the $\alpha\beta$-plane of the triangle T_1 in the uv-plane can be plotted. It is found to have the form shown in Fig. 3.*

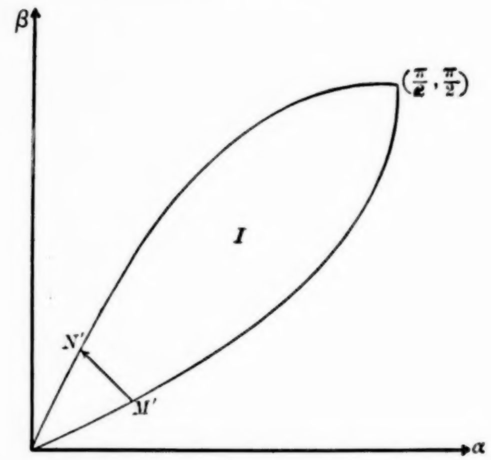

Fig. 3.

As the point (u, v) monotonically describes parallel lines MN of the triangle T_1 in the uv-plane, the corresponding point (α, β) describes monotonically the parallel lines $M'N'$ shown in the region I. It is evident that the correspondence between T_1 and its image I in the $\alpha\beta$-plane is a one-to-one correspondence. One has then the following theorem: *If the two angles α and β are given, for which the point (α, β) lies in the region I, then there exists one, and only one, pair of points A and B on the same arch of the cycloid C for which the triangle ABC has the given angles. If the point (α, β) lies outside of I there is no pair of points A and B which will give the required angles.*

The connection is now readily made with the original problem. *When the two points a and b and the directed lines ac and cb through them are given, there is either one or no cycloid joining a and b, having the tangents ac and cb and containing no cusp on the arc ab. If the angles α and β are given δ is found by means of relation (5), and then α_1 and α_2 can be determined by the equations (7), (8), (9). If α lies between α_1 and α_2 the construction can be made, otherwise not.*

§ 2. Cycloid arcs containing n cusps.

When the extension is made to arcs containing n cusps it will be found convenient to suppose that the point A is on the first arch of the cycloid

* The point $(0, 0)$ of the $\alpha\beta$-plane is the only point of image I to be excluded, the remainder of the boundary being included. This point corresponds to the value $\delta = 0$ and would require that the two points A and B coincide.

to the right of the y-axis. The values of u and v corresponding to the points A and B will always lie in the intervals

$$-\frac{\pi}{2} \leqq u \leqq \frac{\pi}{2}, \quad (2n-1)\frac{\pi}{2} \leqq v \leqq (2n+1)\frac{\pi}{2}.$$

If then α, β, δ are defined as the angles obtained in going from AC to AB, AB to CB and AC to CB through the angle whose absolute value is less than or equal to π, the following expressions for α and β in terms of u, v and φ are obtained

$$(10) \qquad \begin{cases} \alpha = \varphi - u, \\ \beta = v - \varphi - n\pi. \end{cases}$$

Hence the range of values of the angle δ defined by the equation

$$(11) \qquad \delta = \alpha + \beta = v - u - n\pi$$

is readily verified to be from $-\pi$ to $+\pi$. Likewise the equation which defines α is

$$(12) \qquad \tan\alpha = \frac{\sin\delta\sin(u+\delta) - (n\pi+\delta)\sin u}{(n\pi+\delta)\cos u + \sin\delta\cos(u+\delta)},$$

this being analogous to equation (6) of §1.

Denoting the right-hand side of equation (12) by $G(u)$ it is at once seen that its derivative $G'(u)$,

$$G'(u) = \frac{\sin^2\delta - (n\pi+\delta)^2}{[(n\pi+\delta)\cos u + \sin\delta\cos(u+\delta)]^2},$$

is always negative or zero.

For further consideration it will be necessary to make a division of cases according as δ is positive or negative.

Case I. $0 \leqq \delta \leqq \pi$.

In case δ is positive all the admissible values of the point (u, v) lie in the triangle T_{2n} (Fig. 2). From this it follows that for a given constant value of δ between 0 and π the values of u for points on the line $v = u + \delta + n\pi$ in the triangle T_{2n} have the range

$$-\frac{\pi}{2} \leqq u \leqq \frac{\pi}{2} - \delta.$$

In this interval the derivative $G'(u)$ is negative and therefore the function decreasing. Moreover the denominator of $G(u)$ is always positive, since

each of its terms is positive in the given interval. The numerator, on the other hand, vanishes at most once, as is seen from the formula

$$\sin u \sin (u + \delta) - (n\pi + \delta) \sin u$$
$$= (n\pi + \delta - \sin \delta \cos \delta) \cos u \left\{ \frac{\sin^2 \delta}{n\pi + \delta - \sin \delta \cos \delta} - \tan u \right\}.$$

As for the limiting values of $G(u)$ it is easily seen that the expression

$$G\left(-\frac{\pi}{2}\right) = \frac{n\pi + \delta - \sin \delta \cos \delta}{\sin^2 \delta}$$

is always positive for all values of δ in the range $0 \gtreqless \delta \gtreqless \pi$ and becomes infinite when $\delta = 0$, while the expression

$$G\left(\frac{\pi}{2} - \delta\right) = \frac{-(n\pi + \delta) \cos \delta + \sin \delta}{(n\pi + \delta) \sin \delta}$$

is sometimes positive and sometimes negative, according to the value of δ. It has the value $-\infty$ when $\delta = 0$ and the value $+\infty$ for $\delta = \pi$. Hence, *for a given positive value of δ the angle α decreases continuously from a value $\alpha_1 (0 < \alpha_1 \gtreqless \pi/2)$ to a value $\alpha_2 (-\pi/2 \gtreqless \alpha_2 \gtreqless \pi/2)$ Then from the relation $\alpha + \beta = \delta$ it follows that there is but one value of β corresponding to each of these values of α.*

The image of the line $v = u + \delta + n\pi$ in the $\alpha\beta$-plane is the segment of the line $\alpha + \beta = \delta$ between the ordinates α_1 and α_2. The locus of the points (α_1, β_1), (α_2, β_2) for the different values of δ can be plotted by means of the formulae

$$\tan \alpha_1 = \frac{n\pi + \delta - \sin \delta \cos \delta}{\sin^2 \delta}, \tag{13}$$

$$\tan \alpha_2 = \frac{\sin \delta - (n\pi + \delta) \cos \delta}{(n\pi + \delta) \sin \delta}, \tag{14}$$

which satisfy the relation

$$\tan \alpha_2 = \tan (\delta - \alpha_1) = \tan \beta_1. \tag{15}$$

This last formula again indicates symmetry with respect to the bisectors of the first and third quadrants of the plane.

From these last equations it is seen that for any fixed n the curves traced by the points (α_1, β_1), (α_2, β_2) lie in the interior of the isosceles triangle whose vertices are $(-\pi/2, \pi/2)$, $(\pi/2, \pi/2)$, $(\pi/2, -\pi/2)$ and have these three points in common. Moreover, if a comparison of results is made when two particular values of n are substituted in equations (13) and (14) it is seen

that the curves flatten towards the lines $\alpha = \pi/2$, $\beta = \pi/2$ when n increases. In other words the curve defined by these equations for a definite value of n includes the curve for the same equations when n is replaced by $n - 1$. In particular will this be the case for $n = 0$, as is readily verified by comparing equations (13) and (14) with (7) and (8).

Hence, *when the angles α and β of the triangle ABC define a point (α, β) in the region defined by the equations* (13), (14), (15), *there is one and but one pair of points A and B on the first and $(n + 1)$th arches of the cycloid for which ABC has the required angles. Therefore, if any, there is but one cycloid arc containing n cusps and passing through the points a and b in the required directions. If the point (α, β) does not lie in this region there is no solution of the original problem.*

In order to see if a construction is possible when α and β, and therefore δ, are given, it is only necessary to find the values of α_1 amd α_2 from the formulæ (13) and (14) and see if the given value of α lies between these two.

Finally it should be noticed that when δ is positive, any two points a and b which can be joined by an arc containing no cusps can always be joined by an arc containing n cusps, but not conversely. This conclusion follows at once from the preceeding statements regarding the regions in the $\alpha\beta$-plane.

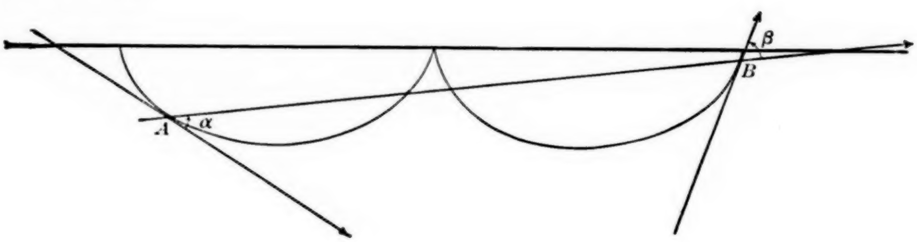

FIG. 4.

As a special example illustrating these results consider the case when $n = 1$. Then, as follows from Fig. 4, it is seen that u and v lie in the intervals

$$-\frac{\pi}{2} \leqq u \leqq \frac{\pi}{2}, \quad \frac{\pi}{2} \leqq v \leqq \frac{3\pi}{2}.$$

Further the relations (10) and (11) now become

$$(10') \qquad \begin{cases} \alpha = \varphi - u, \\ \beta = v - \varphi - \pi, \end{cases}$$

and

$$(11') \qquad \delta = \alpha + \beta = v - u - \pi,$$

respectively, from which it follows that u actually has the range

$$-\frac{\pi}{2} \leqq u \leqq \frac{\pi}{2} - \delta.$$

Finally the formulæ for computing (α_1, β_1) and (α_2, β_2) derived from (13) and (14) for $n = 1$ are

$$\tan\alpha_1 = \frac{\pi + \delta - \sin\delta\cos\delta}{\sin^2\delta}, \tag{13'}$$

$$\tan\alpha_2 = \frac{\sin\delta - (\pi + \delta)\cos\delta}{(\pi + \delta)\sin\delta}. \tag{14'}$$

From these relations table B can be computed.

Table B.

δ	$\alpha_1 = \beta_2$	$\beta_1 = \alpha_2$
0.0	1.571	−1.571
.524	1.493	− .969
.785	1.427	− .642
1.047	1.374	− .327
1.228	1.356	− .134
1.309	1.329	− .020
1.396	1.353	.043
1.571	1.361	.210
2.094	1.414	.680
2.356	1.486	.870
2.618	1.530	1.088
3.142	1.571	1.571

From this table it is found that the image II in the $\alpha\beta$-plane of the triangle T_2 in the uv-plane has the form shown in Fig. 5.

Case II. $-\pi \leqq \delta \leqq 0$.

In case the arc AB is to contain n cusps but δ is negative the admissible region of the uv-plane is the triangle T_{2n+1} (Fig. 2). From the graph of the line $v = u + \delta + n\pi$ it follows that the admissible values of u for a constant value of δ between $-\pi$ and 0 are

$$-\frac{\pi}{2} - \delta \leqq u \leqq \frac{\pi}{2}.$$

In this interval it is known that the derivative $G'(u)$ is always negative and consequently $G(u)$, and therefore α, decreases continuously as u increases. In order to find the admissible values of α we will first discuss its end values when $u = -\pi/2 - \delta$ and $u = \pi/2$.

Consider first the expression for the end value $G(-\pi/2 - \delta)$,

$$G\left(-\frac{\pi}{2} - \delta\right) = \frac{(n\pi + \delta)\cos\delta - \sin\delta}{-(n\pi + \delta)\sin\delta}.$$

Its numerator has the positive derivative $-(n\pi + \delta) \sin \delta$, and assumes the value $-(n-1)\pi$ when $\delta = -\pi$. For further discussion it is now necessary to distinguish two cases according as $n > 1$ or $n = 1$. If $n > 1$ the numerator varies from $-(n-1)\pi$ to $n\pi$ as δ varies from $-\pi$ to 0. But the denominator is always positive, being zero at the end values $-\pi$ and 0. Hence it follows that the expression $G(-\pi/2 - \delta)$ varies from $-\infty$ to $+\infty$ and becomes zero but once for δ between $-\pi$ and 0. If $n = 1$ it is at once seen that $G(-\pi/2 - \delta)$ varies only between 0 and $+\infty$ for this range of values.

Consider next the expression $G(\pi/2)$,

$$G\left(\frac{\pi}{2}\right) = \frac{(n\pi + \delta) - \sin \delta \cos \delta}{\sin^2 \delta}.$$

The numerator of this fraction has the positive derivative $1 - \cos 2\delta$ and varies from $(n-1)\pi$ to $n\pi$ as δ varies from $-\pi$ to 0. So both numerator and denominator of $G(\pi/2)$ are positive, therefore $G(\pi/2)$ itself, and it has the value $+\infty$ for δ equal both $-\pi$ and 0 if $n > 1$. If $n = 1$ the end values of $G(\pi/2)$ are 0 and $+\infty$.

Furthermore $G(u)$ can vanish but once in the interval $-\pi/2 - \delta \gtreqless u \gtreqless \pi/2$, if at all. In order to see this it is only necessary to examine the zeros of the numerator which we will denote by $N(u)$. The derivative of this

$$N'(u) = \sin \delta \cos (u + \delta) - (n\pi + \delta) \cos u$$

is always negative. In fact each term of the derivative is always negative as is readily verified when it is remembered that $u + \delta = v - n\pi$.

If $n = 1$ this numerator $N(u)$ has the positive value

$$(\pi + \delta) \cos \delta - \sin \delta$$

when $u = -\pi/2 - \delta$ and the negative value

$$-(\pi + \delta) + \sin \delta \cos \delta$$

when $u = \pi/2$. Hence there is one and only one value of u making $G(u)$ vanish. From this it follows that there is only one value of u making $G(u)$ infinite. For, as has been shown, $G(u)$ is a decreasing function of u such that both end values are positive, and there is one point of vanishing. Hence $G(u)$ becomes negative and must therefore become infinite if its end value is positive.

On the other hand, if $n > 1$, it is found that, if the numerator is considered as a function of δ when u is replaced by $-\pi/2 - \delta$, it then has a positive derivative and changes continuously from a negative to a positive

value as δ varies from $-\pi$ to 0. Thus it happens that, for some values of δ, the function $G(u)$ does not become zero but starts from a negative value and decreases through infinity to a positive value $G(\pi/2)$.

Hence the following statements can be made: *If $n = 1$ the angle α decreases for a given δ from a value $\alpha_1 (0 \leqq \alpha_1 \leqq \pi/2)$ through zero and $-\pi/2$ to a value α_2 $(-\pi \leqq \alpha_2 \leqq -\pi/2)$. To each of these values there corresponds*

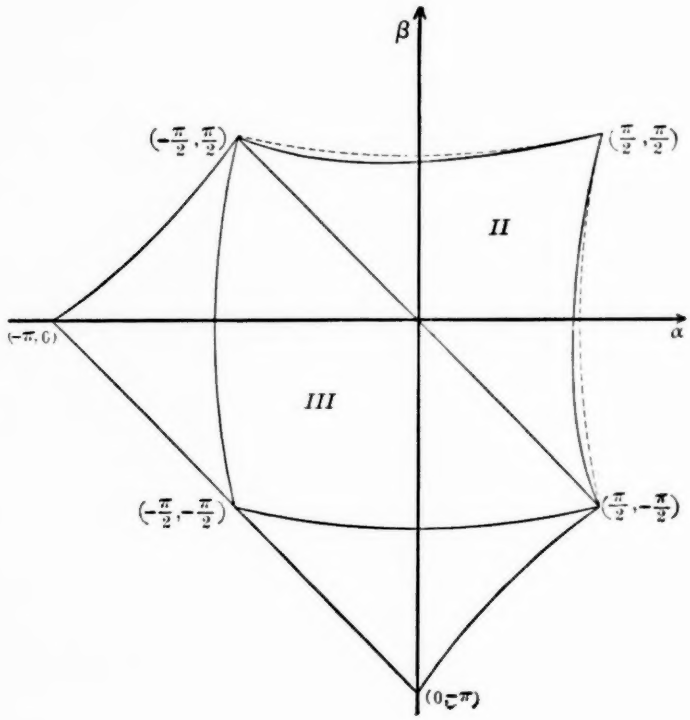

FIG. 5.

but one value of β. If $n > 1$ the angle α decreases from a value $\alpha_1 (-\pi/2 \leqq \alpha_1 \leqq \pi/2)$ to a value α_2 less than $-\pi/2$.

The points (α_1, β_1), (α_2, β_2) defining the admissible region of the $\alpha\beta$-plane are then given by the formulæ

$$\tan \alpha_1 = \frac{\sin \delta - (\pi + \delta) \cos \delta}{(\pi + \delta) \sin \delta}, \tag{13''}$$

$$\tan \alpha_2 = \frac{\pi + \delta - \sin \delta \cos \delta}{\sin^2 \delta}, \tag{14''}$$

$$\tan \alpha_2 = \tan (\delta - \alpha_1), \tag{15''}$$

in case $n = 1$; and by the formulæ

$$\tan \alpha_1 = \frac{\sin \delta - (n\pi + \delta) \cos \delta}{(n\pi + \delta) \sin \delta}, \tag{13'''}$$

$$\tan \alpha_2 = \frac{n\pi + \delta - \sin \delta \cos \delta}{\sin^2 \delta}, \tag{14'''}$$

$$\tan \alpha_2 = \tan (\delta - \alpha_1) \tag{15'''}$$

in case $n > 1$.

From the formulæ (13''), (14''), (15'') the table C can be computed and a continuous graph joining the points $(0, -\pi)$, $(\pi/2, -\pi/2)$ and $(-\pi, 0)$, $(-\pi/2; \pi/2)$ is found* as shown in III, Fig. 5.

Table C.

δ	$\alpha_1 = \beta_2$	$\alpha_2 = \beta_1$
0.000	1.571	−1.571
−0.524	1.129	−1.653
−0.785	.960	−1.745
−1.047	.812	−1.859
−1.571	.567	−2.138
−2.094	.429	−2.523
−2.356	.268	−2.624
−2.618	.176	−2.794
−3.142	.000	−3.142

As an illustration of the case $n > 1$ let n have the particular value $n = 2$. The formulæ (13'''), (14'''), (15''') then allow us to compute the tables D from which follows the continuous graph joining the points $(-\pi/2, -\pi/2)$, $(-\pi/2, \pi/2)$ and $(-\pi/2, -\pi/2)$, $(\pi/2, -\pi/2)$ as shown in III, Fig. 5.

Table D.

δ	$\alpha_1 = \beta_2$	$\alpha_2 = \beta_1$
−0.000	1.571	−1.571
−0.524	1.083	−1.607
−0.785	0.869	−1.654
−1.047	0.653	−1.700
−1.571	0.208	−1.779
−2.094	−0.333	−1.761
−2.356	−0.639	−1.717
−2.618	−0.970	−1.648
−3.142	−1.571	−1.571

By a comparison of the preceding formulæ it is seen that, if $n > 1$, then, for an increasing number of cusps, the curves defining the boundary

* In case $n = 1$ and $\delta = -\pi$ the points A and B must coincide at the cusp, as is readily verified geometrically. Hence this value of δ must be excluded.

region of the $\alpha\beta$-plane flatten towards the lines

$$\beta = -\frac{\pi}{2}, \quad \alpha = -\frac{\pi}{2}.$$

We therefore conclude that *when the angles* α *and* β *of the triangle* ABC *define for a negative* δ *a point* (α, β) *in the region* III *corresponding to the given number of cusps, n, then there is one, and but one, pair of values* (u, v) *in the region* T_{2n+1} (*Fig.* 2) *and therefore one and but one pair of points* A *and* B *in the first and* $(n + 1)$*th arches of the cycloid for which the triangle has the required angles. Hence there is one and but one cycloid arc containing n cusps and passing through the points a and b in the directions prescribed by the original problem. If for a given n the point* (α, β) *lies outside its region in* III *no solution of the original problem exists.*

Finally it follows from the conclusions regarding the regions in the $\alpha\beta$-plane that for a negative δ, *any two points a and b which can be joined by an arc containing n cusps can always be joined by one containing* $n - 1$ *cusps, but not conversely.*

Sheffield Scientific School,
Yale University.

THEOREMS ON REDUCIBLE QUANTICS.

By O. E. Glenn.

§1. Introduction.

The content of §2 of the following series of theorems comprises a generalization of the Gauss-Eisenstein* theorem on the arithmetical condition for the irreducibility of a binary form in the absolute field, and some new results growing out of the principle of this generalization. We can state this theorem as follows:

A necessary condition that the binary form with integral coefficients,

$$f(x_1, x_2) = c_0 x_1^m + c_1 x_1^{m-1} x_2 + \cdots + c_m x_2^m,$$

all of whose coefficients except c_0 are divisible by a prime q, should be reducible in the absolute field $R(1)$ is that

$$c_m \equiv 0 \pmod{q^2}. \tag{1}$$

The extension of this theorem to the case of p-ary forms given in §2 is more or less direct as to the method, but the results seem of interest in themselves aside from later applications, for the criteria for reducibility become sharper as p increases. This is simply due to the fact that the number of necessary conditions corresponding to (1) is, in the general case, an increasing function of p.

In §3 we give, with certain amplifications, a new analytical theory of the linear factorability of a p-ary form. Perhaps the most unique feature of this theory is its complete generality with reference to multiple factors, and the cases where pairs of factors, though not multiple, are nevertheless not term-wise distinct. None of the theories published heretofore are general in this regard. This development leads to the important theorem that if a linearly factorable p-ary form has its leading binary form linearly reducible in a given field the p-ary form is reducible in the same field [Th. 4, 5].

In §4 some of the results of the previous sections are combined.

* Gauss, Disquisitiones Arithmeticæ, § 42. Eisenstein: "Ueber die Irreductibilität u. s. w. der Lemniscatengleichung." Journal für Mathem., Vol. 39, p. 160.

Königsberger, "Ueber den Eisenstein'schen Satz von Irreductibilität algebraischer Gleichungen." Journal für Mathem., Vol. 115, p. 53. See also Netto, Vorlesungen ueber Algebra, Vol. 1, p. 51.

Kronecker, "Grundzüge einer arithmetischen Theorie der algebraischen Grössen." Journal für Mathem., Vol. 92 (1882).

§2. Generalization of Eisenstein's theorem, etc.

Let us represent the general p-ary form f with integral coefficients, of order $N = m + n$, by means of the following notation:

$$(2)\qquad f_{pN} = \sum_{j_1, \dots, j_p = 0}^{N} c_{j_1 j_2 \dots j_p} x_1^{j_1} x_2^{j_2} \cdots x_p^{j_p} \quad (j_1 + j_2 + \cdots + j_p = N;\ m \geqq n),$$

no numerical factors being involved with the c's. Assume that f_{pN} is reducible in the absolute field, so that for some set of values (m, n) we will have a relation of the form

$$f_{pN} = \left(\sum_{k_1, \dots, k_p = 0}^{m} a_{k_1 k_2 \dots k_p} x_1^{k_1} x_2^{k_2} \cdots x_p^{k_p}\right)\left(\sum_{l_1, \dots, l_p = 0}^{n} b_{l_1 l_2 \dots l_p} x_1^{l_1} x_2^{l_2} \cdots x_p^{l_p}\right)$$
$$(k_1 + k_2 + \cdots + k_p = m;\ l_1 + l_2 + \cdots + l_p = n),$$

where the a's and b's are all integers.* Then

$$(3)\qquad c_{j_1 j_2 \dots j_p} = \sum_{\substack{\kappa = 0, \dots, m \\ \lambda = 0, \dots, n}} a_{\kappa_1 \kappa_2 \dots \kappa_p} b_{\lambda_1 \lambda_2 \dots \lambda_p},$$

where

$$(4)\qquad \sum_r \kappa_r = m,\ \sum_r \lambda_r = n,\ \sum_r j_r = \sum_r \kappa_r + \sum_r \lambda_r,\ j_r = \kappa_r + \lambda_r \quad (r = 1, 2, \cdots, p).$$

It is necessary to call particular attention to certain features of this notation, as the proofs of a number of theorems depend upon arithmetical properties of the bilinear expressions (3). We note first that these bilinear expressions can be written down immediately as soon as m and n are known. For the subscripts of the c's, i. e., the sets of numbers $(j_1, j_2, \cdots, j_p)$ are the partitions of N into p parts, repetitions being permitted, each part having the range $0, 1, \cdots, m + n$ inclusive. Having thus written the complete set of c's we obtain the value of any one, as $c_{j_1 j_2 \dots j_p}$, in terms of the a's and b's in the following manner: write all the partitions of j_r $(r = 1, 2, \cdots, p)$ into two parts (κ_r, λ_r), κ_r having the range $0, 1, \cdots, m$ inclusive, and λ_r the range $0, 1, \cdots, n$ inclusive. This gives all the double sets $[(\kappa_1, \kappa_2, \cdots, \kappa_p), (\lambda_1, \lambda_2, \cdots, \lambda_p)]$ which satisfy (4), and hence all of the terms of the summation on the right hand side of (3).

Secondly, with reference to two sets of p numbers $(\kappa_1, \kappa_2, \cdots, \kappa_p)$, $(\lambda_1, \lambda_2, \cdots, \lambda_p)$, we say that the sets occur in *normal order* if the set first to show, when read from right to left, a number greater than the number in the corresponding position in the other set occurs farthest to the right.

In all that follows we may (and do) assume that the terms of each bilinear expression (3) have been arranged so that the subscripts of the b's are in

* If a p-ary form with integral coefficients is reducible in the absolute field into two factors, it is reducible into two factors having integral coefficients. Cf. Gauss: Loc cit., and the proof of theorem (1) below. The details of the proof of this lemma are omitted for brevity.

normal order.* Then, by virtue of (4), the subscripts of the a's occur in the exact reverse of the normal order. For convenience we also assume that the c's are arranged in a vertical column with subscripts arranged in normal order, the upper end of the column corresponding to the left hand end of a normal arrangement. There are p bilinear expressions which consist of a single term. These are

$$(5)\qquad c_{\overbrace{0\ldots0}^{p-j}\,m+n\,\overbrace{0\ldots0}^{j-1}} = a_{\overbrace{0\ldots0}^{p-j}\,m\,\overbrace{0\ldots0}^{j-1}}\, b_{\overbrace{0\ldots0}^{p-j}\,n\,\overbrace{0\ldots0}^{j-1}} \qquad (j = p, p-1, \cdots, 2, 1)$$

In the above normal arrangement of the c's the first of the c's (5) ($j = p$) is the first in the column and the last ($j = 1$) is the last in the column.

Consider now a prime number q which divides $c_{00\cdots0m+n}$. It will divide a group of the c's (5) and fail to divide the rest. By a suitable re-adjustment of the notation of $x_1, \cdots, x_p$, we can bring those not divisible by q to the top of the column, that is, secure a normal arrangement of the c's such that

$$(6)\qquad \begin{aligned} &c_{\overbrace{0\ldots0}^{p-j}\,m+n\,\overbrace{0\ldots0}^{j-1}} \equiv 0 \pmod{q} \quad (j = 1, 2, \cdots, \mu, \cdots, \mu+\nu) \\ &c_{\overbrace{0\ldots0}^{p-j}\,m+n\,\overbrace{0\ldots0}^{j-1}} \not\equiv 0 \pmod{q} \quad (j = \mu+\nu+1, \mu+\nu+2, \cdots, p), \end{aligned}$$

namely

$$(7)\qquad \begin{aligned} &c_{m+n00\cdots0} = a_{m00\cdots0}\, b_{n00\cdots0}, \\ &c_{m+n-110\cdots0} = a_{m-110\cdots0}\, b_{n00\cdots0} + a_{m00\cdots0}\, b_{n-110\cdots0}, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &c_{\overbrace{0\ldots\ldots0}^{p-\mu-\nu-1}\,m+n\,\overbrace{0\ldots0}^{\mu+\nu}} = a_{\overbrace{0\ldots\ldots0}^{p-\mu-\nu-1}\,m\,\overbrace{0\ldots0}^{\mu+\nu}}\, b_{\overbrace{0\ldots\ldots0}^{p-\mu-\nu-1}\,n\,\overbrace{0\ldots0}^{\mu+\nu}}, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &c_{j_1 j_2\ldots j_p} = \cdots \\ &\cdot\ \cdot \\ &c_{\overbrace{0\ldots0}^{p-i-1}\,1\,m+n-1\,\overbrace{0\ldots0}^{i-1}} = a_{\overbrace{0\ldots0}^{p-i}\,m\,\overbrace{0\ldots0}^{i-1}}\, b_{\overbrace{0\ldots\ldots0}^{p-i-1}\,1\,n-1\,\overbrace{0\ldots0}^{i-1}} + a_{\overbrace{0\ldots0}^{p-i-1}\,1\,m-1\,\overbrace{0\ldots0}^{i-1}}\, b_{\overbrace{0\ldots0}^{p-i}\,n\,\overbrace{0\ldots\ldots0}^{i-1}}, \\ &c_{\overbrace{0\ldots0}^{p-i}\,m+n\,\overbrace{0\ldots0}^{i-1}} = a_{\overbrace{0\ldots0}^{p-i}\,m\,\overbrace{0\ldots0}^{i-1}}\, b_{\overbrace{0\ldots0}^{p-i}\,n\,\overbrace{0\ldots0}^{i-1}}, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &c_{\overbrace{0\ldots0}^{p-\mu}\,m+n\,\overbrace{0\ldots0}^{\mu-1}} = a_{\overbrace{0\ldots0}^{p-\mu}\,m\,\overbrace{0\ldots0}^{\mu-1}}\, b_{\overbrace{0\ldots0}^{p-\mu}\,n\,\overbrace{0\ldots0}^{\mu-1}}, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ &c_{00\cdots0m+n} = a_{00\cdots0m}\, b_{00\cdots0n}. \end{aligned}$$

* The b's would then be precisely in the order in which they would occur as the coefficients of a p-ary n-ic arranged according to x_1 as leading letter, x_2 as the letter next in order, etc.

Theorem 1: *A set of necessary conditions that a form f, all of whose coefficients within the interval*

$$I = \{c_{\overbrace{0\ldots0}^{p-\mu}m+n\overbrace{0\ldots0}^{\mu-1}}, \cdots, {}^*c_{\overbrace{0\ldots\ldots0}^{p-\mu-\nu-1}m+n\overbrace{0\ldots0}^{\mu+\nu}}\}^*$$

are divisible by a prime q should be reducible in the absolute field is given by

$$c_{\overbrace{0\ldots0}^{p-i}m+n\overbrace{0\ldots0}^{i-1}} \equiv 0 \ (mod\ q^2) \quad (i = \mu, \mu+1, \cdots, \mu+\nu).$$

In proof assume [1] that, for a particular i of the set $\mu, \cdots, \mu+\nu$

$$c_{\overbrace{0\ldots0}^{p-i}m+n\overbrace{0\ldots0}^{i-1}} \not\equiv 0 \ (\text{mod } q^2).$$

Then one of the numbers

$$a_{\overbrace{0\ldots0}^{p-i}m\overbrace{0\ldots0}^{i-1}}, \quad b_{\overbrace{0\ldots0}^{p-i}n\overbrace{0\ldots0}^{i-1}},$$

say [2] the latter, is incongruent to zero modulo q. There is one and only one bilinear expression $(c_{j_1 j_2 \ldots j_p})$ which contains the term

$$a_{\overbrace{0\ldots\ldots0}^{p-\mu-\nu-1}m\overbrace{0\ldots0}^{\mu+\nu}} b_{\overbrace{0\ldots0}^{p-i}n\overbrace{0\ldots0}^{i-1}} \ (= t).$$

Since the c's and b's are arranged in normal order t is the last term in $c_{j_1\ldots j_p}$. By assumption [2] the

$$c_{\overbrace{0\ldots0}^{p-i-1}1m+n-1\overbrace{0\ldots0}^{i-1}}$$

in (7) gives at once from the hypothesis of he theorem

$$a_{\overbrace{0\ldots0}^{p-i-1}1m-1\overbrace{0\ldots0}^{i-1}} \equiv \ (\text{mod } q).$$

By examining in turn the consecutive c's from

$$c_{\overbrace{0\ldots0}^{p-i}m+n\overbrace{0\ldots0}^{i-1}}$$

upward to $c_{j_1 j_2 \ldots j_p}$ we prove in the same manner that all the a's occurring in t, except

$$a_{\overbrace{0\ldots\ldots0}^{p-\mu-\nu-1}m\overbrace{0\ldots0}^{\mu+\nu}},$$

are divisible by q. Hence, since the $c_{j_1 j_2 \ldots j_p}$ is in I, we have

$$a_{\overbrace{0\ldots\ldots0}^{p-\mu-\nu-1}m\overbrace{0\ldots0}^{\mu+\nu}} \equiv 0 \ (\text{mod } q),$$

* By this notation we indicate that the starred number is not included in the interval. It only shows the upper limit of the interval.

contrary to the hypothesis of the theorem. Hence

$$c_{\overbrace{0\ldots0}^{p-i}\,m+n\,\overbrace{0\ldots0}^{i-1}} \equiv 0 \pmod{q^2}$$

or else the form f is irreducible.

Corollary 1: If $p = 2$ all of the coefficients of the form after the first are included in I, and we have Eisenstein's theorem (§ 1).

Corollary 2: If

$$c_{\overbrace{0\ldots0}^{p-i}\,m+n\,\overbrace{0\ldots0}^{i-1}} \not\equiv 0 \pmod{q^3},$$

then

$$a_{\overbrace{0\ldots0}^{p-i}\,m\,\overbrace{0\ldots0}^{i-1}} = q\alpha^0_{\overbrace{0\ldots0}^{p-i}\,m\,\overbrace{0\ldots0}^{i-1}}, \quad b_{\overbrace{0\ldots0}^{p-i}\,n\,\overbrace{0\ldots0}^{i-1}} = q\beta^0_{\overbrace{0\ldots0}^{p-i}\,n\,\overbrace{0\ldots0}^{i-1}}.*$$

Theorem 2: *If the* τ† *consecutive coefficients forming the sub-interval of* I

$$i_1 = [c_{\overbrace{0\ldots0}^{p-\mu}\,m+n\,\overbrace{0\ldots0}^{\mu-1}} = q^2\gamma^0_{\overbrace{0\ldots0}^{p-\mu}\,m+n\,\overbrace{0\ldots0}^{\mu-1}},\ c_{\overbrace{0\ldots0}^{p-\mu-1}\,1\,m+n-1\,\overbrace{0\ldots0}^{\mu-1}},\ \cdots]$$

are divisible by q^2 *and the remaining coefficients within the interval* I *are divisible by* q *to the first power; and if the order* $N = m + n$ *of* f *satisfies the condition*

$$(8)\quad \binom{\tfrac{1}{2}N+p-\mu-1}{p-\mu} + \binom{\tfrac{1}{2}N+p-\mu-2}{p-\mu-1} + \cdots + \binom{\tfrac{1}{2}N+p-\mu-\nu-1}{p-\mu-\nu} \geqq \tau,$$

then the order n *of a possible factor of* f *must satisfy the condition*

$$(9)\quad \binom{n+p-\mu-1}{p-\mu} + \binom{n+p-\mu-2}{p-\mu-1} + \cdots + \binom{n+p-\mu-\nu-1}{p-\mu-\nu} \geqq \tau.$$

Since

$$a_{\overbrace{0\ldots0}^{p-\mu}\,m\,\overbrace{0\ldots0}^{\mu-1}} \quad\text{and}\quad b_{\overbrace{0\ldots0}^{p-\mu}\,n\,\overbrace{0\ldots0}^{\mu-1}}$$

are, by Corollary 2, both divisible by q we can assume (that which will always be true for $\sigma = 1$ at least) that the like is true for all of the σ consecutive a numbers forming the interval

$$J = \{a_{\overbrace{0\ldots0}^{p-\mu}\,m\,\overbrace{0\ldots0}^{\mu-1}} = q\alpha^0_{\overbrace{0\ldots0}^{p-\mu}\,m\,\overbrace{0\ldots0}^{\mu-1}},\ a_{\overbrace{0\ldots0}^{p-\mu-1}\,1\,m-1\,\overbrace{0\ldots0}^{\mu-1}},\ \cdots\} \qquad (\sigma < \tau),$$

and of all of the σ consecutive b numbers forming the interval

$$K = \{b_{\overbrace{0\ldots0}^{p-\mu}\,n\,\overbrace{0\ldots0}^{\mu-1}} = q\beta^0_{\overbrace{0\ldots0}^{p-\mu}\,n\,\overbrace{0\ldots0}^{\mu-1}},\ b_{\overbrace{0\ldots0}^{p-\mu-1}\,1\,n-1\,\overbrace{0\ldots0}^{\mu-1}},\ \cdots\}.$$

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* By a zero superscript we indicate that a number is prime to q.

† We shall refer to an interval containing τ elements as being of extent τ.

Then the σ consecutive c numbers forming the sub-interval of I

$$i_2 = \{c_{\overbrace{0\ldots0}^{p-\mu}m+n\overbrace{0\ldots0}^{\mu-1}},\ c_{\overbrace{0\ldots0}^{p-\mu-1}1m+n-1\overbrace{0\ldots0}^{\mu-1}},\ \cdots\}$$

have all their terms individually divisible by q^2, while the $\sigma + 1$th c, say for brevity $c_{(\sigma+1)}$, has the $\sigma + 1$th a, say $a_{(\sigma+1)}$, for a factor of its last term and the $\sigma + 1$th b, say $b_{(\sigma+1)}$, as a factor of its first term. Moreover since the b's are in normal order and the a's in the reverse order, these end terms are divisible each by q only, whereas the intermediate terms are all divisible by q^2. Thus we have after depressing the modulus by the factor q

$$a_{(\sigma+1)}\beta^\circ + \alpha^\circ b_{(\sigma+1)} \equiv 0 \pmod{q}.$$

Now there is a number c *within* the interval I which contains the term $a_{(\sigma+1)}b_{(\sigma+1)}$. Obviously also every other term of this c is divisible by q, so that we have

$$a_{(\sigma+1)}b_{(\sigma+1)} \equiv 0 \pmod{q}.$$

Hence

$$a_{(\sigma+1)} \equiv b_{(\sigma+1)} \equiv 0 \pmod{q}.$$

By repeating this argument we can increase the extent of the intervals J, K, i_2 until the extent of each is τ, provided $b_{(\tau)}$ falls within the interval formed by consecutive b's

$$L = \{b_{\overbrace{0\ldots0}^{p-\mu}n\overbrace{0\ldots0}^{\mu-1}},\ \cdots,\ {}^*b_{\overbrace{0\ldots\ldots0}^{p-\mu-\nu-1}n\overbrace{0\ldots0}^{\mu+\nu}}\}.$$

If $b_{(\tau)}$ does not we will, on the other hand, be led to the contradiction of hypothesis (6)

$$b_{\overbrace{0\ldots\ldots0}^{p-\mu-\nu-1}n\overbrace{0\ldots0}^{\mu+\nu}} \equiv 0 \pmod{q}.$$

Hence it is necessary that $b_{(\tau)}$ be contained in L. But the extent of L equals the left hand member of (9), and therefore the order n of a possible factor of f must satisfy (9).

The maximum value of n is $\frac{1}{2}N$. Hence if (8) is assumed, factors of f are possible having orders ranging from the minimum n satisfying (9) up to $\frac{1}{2}N$. This completes the proof.

Corollary 3. *If N is odd and*

$$\tau > \binom{\frac{1}{2}(N-1) + p - \mu - 1}{p - \mu} + \cdots + \binom{\frac{1}{2}(N-1) + p - \mu - \nu - 1}{p - \mu - \nu},$$

then the form f is irreducible. If N is even and

$$\tau \geqq \binom{\frac{1}{2}N + p - \mu - 1}{p - \mu} + \cdots + \binom{\frac{1}{2}N + p - \mu - \nu - 1}{p - \mu - \nu},$$

then f is irreducible unless it presents the one possible exceptional case of a form of even order N equal to the product of two forms of the same order $\frac{1}{2}N$.

In a particular case, say $N = 4$, $p = 4$, this corollary specifies an extensive class of irreducible forms, as in the following table:

μ	ν	Value τ; f irreducible unless = product of two quaternary quadratics
3	0	$\tau \geqq 2$
2	0	$\tau \geqq 3$
1	0	$\tau \geqq 4$
2	1	$\tau \geqq 5$
1	1	$\tau \geqq 7$
1	2	$\tau \geqq 9$

§ 3. On the resolution of p-ary forms. Field of reducibility.

If we take the product of r binary forms in as many non-homogeneous variables, viz.

$$x_i^{\nu_i} + a_1^{(i)} x_i^{\nu_i - 1} + a_2^{(i)} x_i^{\nu_i - 2} + \cdots + a_{\nu_i}^{(i)} \qquad (i = 1, 2, \cdots, r),$$

and express the result in the form $\sum_{j=0}^{\nu_1+\nu_2+\cdots+\nu_r} \varphi_j$, where φ_j embraces all of the terms of the product of order

$$= \nu_1 + \nu_2 + \cdots + \nu_r - j,$$

we get

$$\prod_{k=1}^{r} x_k^{\nu_k} + \left(a_1^{(i)} \prod_{k=1}^{r} x_k x_i^{-1} + \textit{terms in } x_i \right) \prod_{k=1}^{r} x_k^{\nu_k - 1}$$

$$(10) \qquad + \left(a_2^{(i)} \prod_{k=1}^{r} x_k^2 x_i^{-2} + \textit{terms in } x_i \right) \prod_{k=1}^{r} x_k^{\nu_k - 2} + \cdots$$

$$+ \left(a_{\nu_i}^{(i)} \prod_{k=1}^{r} x_k^{\nu_i} x_i^{-\nu_i} + \textit{terms in } x_i \right) \prod_{k=1}^{r} x_k^{\nu_k - \nu_i} + \varphi_{\nu_i + 1} + \cdots + \varphi_{\nu_1 + \nu_2 + \cdots + \nu_r}.$$

In case ν_i is not one of the least of the orders ν, negative exponents will occur on the outside of some of the latter parentheses in this formula. When they do occur the corresponding x's with their negative exponents are to be multiplied into the terms in the parenthesis. If negative exponents still remain within the parenthesis when a general f_{r+1m} is arranged according to this formula the terms containing them are to be deleted. The leading terms will never be deleted by this process, however.

From

$$\varphi_j = \left(a_j^{(i)} \prod_{k=1}^{r} x_k^j x_i^{-j} + \textit{terms in } x_i \right) \prod_{k=1}^{r} x_k^{\nu_k - j}$$

we get at once by differentiation

$$(11)\qquad \left\{\frac{\partial^{\nu_i-j}\varphi_j}{\partial x_i^{\nu_i-j}} \Big/ \frac{\partial^{\nu_i}\varphi_0}{\partial x_i^{\nu_i}}\right\}_{x_i=0} = \frac{(\nu_i-j)\,!}{\nu_i\,!}a_j^{(i)}, \quad \begin{pmatrix} j=0,1,\cdots,\nu_i \\ i=1,2,\cdots,r \end{pmatrix}.$$

Accordingly we have

Theorem 3: *Being given an $r+1$-ary form f known to be the product of r polynomials, the single variable in each one of which is one of the r distinct nonhomogeneous variables of the form, we can obtain the coefficients of these polynomials and therefore the polynomial factors of the form f themselves by differentiation according to formula (11). The linear factors of f are then obtained by factorization of the constituent polynomials.*

Let us now write the general ternary form f_{3m} under the notation

$$f_{3m} = x_2^m\varphi_{0\,x_1 x_2} + x_2^{m-1}\varphi_{1\,x_1 x_2}x_3 + x_2^{m-2}\varphi_{2\,x_1 x_2}x_3^2 + \cdots + \varphi_{m\,x_1 x_2}x_3^m,$$

where

$$(12)\qquad \begin{aligned} x_2^{m-r}\varphi_{r\,x_1 x_2} \equiv C_{m-r,0,r}x_1^{m-r} + C_{m-r-1,1,r}x_1^{m-r-1}x_2 + C_{m-r-2,2,r}x_1^{m-r-2}x_2^2 \\ + \cdots + C_{0m-rr}x_2^{m-r}. \end{aligned}$$

Assume that it is linearly factorable. Then the (x_1, x_2) terms of its factors will be furnished by the linear factors of the binary form $x_2^m\varphi_{0\,x_1/x_2}$. Let the factors of the latter be assumed to be known and let them be

$$x_1 + r_i x_2 \qquad (i=1,2,\cdots,t)(c_{m00}=1)^*$$

where $x_1 + r_i x_2$ is of multiplicity α_i, and $\alpha_1+\alpha_2+\cdots+\alpha_t = m$.

Corollary 4: *The ternary form f_{3m} can be factored into factors of the respective orders $\alpha_1, \alpha_2, \cdots, \alpha_t$, which are rational and integral in the coefficients of the form f_{3m} itself on the one hand, and in the quantities r_i on the other, linear in the coefficients; according to the formula,*†

$$(13)\qquad \begin{aligned} f_{3m} = \prod_{i=1}^{t}\Bigg[&\frac{\partial^{\alpha_i}\varphi_{0-r_i}}{\partial r_i^{\alpha_i}}x^{\alpha_i} - \alpha_i\frac{\partial^{\alpha_i-1}\varphi_{1-r_i}}{\partial r_i^{\alpha_i-1}}x^{\alpha_i-1}y_i + \alpha_i(\alpha_i-1)\frac{\partial^{\alpha_i-2}\varphi_{2-r_i}}{\partial r_i^{\alpha_i-2}}x^{\alpha_i-2}y_i^2 - \cdots \\ &+ (-1)^{\alpha_i}\alpha_i\,!\,\varphi_{\alpha_i-r_i}y_i^{\alpha_i}\Bigg], \quad x_1 + r_i x_2/x_3 = \frac{x}{y_i} \quad (i=1,2,\cdots,t). \end{aligned}$$

To prove this we observe that if we multiply together the members of the group of α_i assumed factors of f_{3m}

$$x_1 + r_i x_2 + s_{ij}x_3 \qquad (j=1,2,\cdots,\alpha_i),$$

* This implies that the term x_1^m certainly occurs. It may be necessary to transform f_{3m} linearly to secure this.

† A constant factor has been neglected on the left to secure symmetry. See proof of corollary below.

and in the result set

$$x_1 + r_i x_2 / x_3 = x / y_i \qquad (i = 1, 2, \cdots, t),$$

we get t binary forms

$$x^{\alpha_i} + a_1^{(i)} x^{\alpha_i - 1} y_i + \cdots + a_{\alpha_i}^{(i)} y_i^{\alpha_i} \qquad (i = 1, 2, \cdots, t).$$

Hence f_{3m} is the product of t binary forms and is a case of f in theorem (3). Hence corollary 4 follows directly from (11) when we put $\nu_i = \alpha_i$, $x_i = x / y_i$, $\varphi_j = \varphi_{j\, x_1/x_2}$, $r = t$, and notice that when $x_i = x / y_i = 0$, then $x_1 / x_2 = -r_i$.

Theorem 4: *If the leading binary form $\varphi_{0\, x_1/x_2}$ of a completely reducible ternary form is completely* reducible in a given field the corresponding ternary form f_{3m} is reducible in the same field. It is necessarily completely reducible if the factors of $\varphi_{0\, x_1/x_2}$ are all distinct. It may or may not be completely reducible in this field if some factors of $\varphi_{0\, x_1/x_2}$ are multiple.*

The terms of the general p-ary form f_{pm} [see (2)] can be arranged into the following scheme:

$$\begin{aligned} f_{pm} = {} & x_2^m \varphi_{0\, x_1/x_2} \\ & + x_2^{m-1} x_3 \varphi_{1\, x_1/x_2}^{(1)} + x_2^{m-2} x_3^2 \varphi_{2\, x_1/x_2}^{(1)} + \cdots + x_3^m \varphi_{m\, x_1/x_2}^{(1)} \\ & + x_2^{m-1} x_4 \varphi_{1\, x_1/x_2}^{(2)} + x_2^{m-2} x_4^2 \varphi_{2\, x_1/x_2}^{(2)} + \cdots + x_4^m \varphi_{m\, x_1/x_2}^{(2)} \\ & \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & + x_2^{m-1} x_p \varphi_{1\, x_1/x_2}^{(p-2)} + x_2^{m-2} x_p^2 \varphi_{2\, x_1/x_2}^{(p-2)} + \cdots + x_p^m \varphi_{m\, x_1/x_2}^{(p-2)} \\ & + \Psi_{pm}, \end{aligned}$$

where the typical ternary form

$$\begin{aligned} X^{(j)} = {} & x_2^m \varphi_{0\, x_1/x_2} + x_2^{m-1} x_{j+2} \varphi_{1\, x_1/x_2}^{(j)} \\ & + x_2^{m-2} x_{j+2}^2 \varphi_{2\, x_1/x_2}^{(j)} + \cdots + x_{j+2}^m \varphi_{m\, x_1/x_2}^{(j)} \end{aligned} \tag{14}$$

is given by a notation analogous to (12), viz.,

$$\begin{aligned} x_2^{m-k} \varphi_{k\, x_1/x_2}^{(j)} = {} & C_{m-k\,0\,\overbrace{0\ldots0}^{j-1}k\,\overbrace{0\ldots0}^{p-j-2}} x_1^{m-k} + C_{m-k-1\,1\,\overbrace{0\ldots}^{j-1}k\,\overbrace{0\ldots0}^{p-j-2}} x_1^{m-k-1} x_2 + \cdots \\ & + C_{m-k-h\,h\,\overbrace{0\ldots0}^{j-1}\,k\,\overbrace{0\ldots0}^{p-j-2}} x_1^{m-k-h} x_2^h + \cdots + C_{0\,m-k\,\overbrace{0\ldots0}^{j-1}\,k\,\overbrace{0\ldots0}^{p-j-2}} x_2^{m-k}. \end{aligned}$$

Of course Ψ_{pm} is simply the aggregate of all terms of f_{pm} which are not included in the ternary summands $X^{(j)}$.

Assume that the factors of $x_2^m \varphi_{0\, x_1/x_2}$ are $x_1 + r_i x_2$, as above, where $i = 1, 2, \cdots, t$. Then the typical form $X^{(j)}$ can be resolved by formula (13), and becomes after an unimportant numerical factor is neglected on

* We shall call a form *completely reducible* in a field when it is resolvable into linear factors in that field.

the left

$$X^{(j)} = \prod_{i=1}^{t}\left[\frac{\partial^{\alpha_i}\varphi_{0-r_i}}{\partial r_i^{\alpha_i}} x^{\alpha_i} - \alpha_i \frac{\partial^{\alpha_i-1}\varphi_{1-r_i}{}^{(j)}}{\partial r_i^{\alpha_i-1}} x^{\alpha_i-1}y_i + \cdots + (-1)^{\alpha_i}\alpha_i\,!\varphi_{\alpha_i-r_i}{}^{(j)}y_i^{\alpha_i}\right] \tag{15}$$

$$(x_1 + r_ix_2 \,/\, x_{j+2} = x \,/\, y_i).$$

By resolving into linear factors the binary forms which constitute the rational factors of this form $X^{(j)}$ we get the coefficients r_{ij+2k} of the linear factors

$$x_1 + r_ix_2 + r_{ij+2k}x_{j+2} \qquad \begin{pmatrix} i = 1, 2, \cdots, t \\ k = 1, 2, \cdots, \alpha_i \end{pmatrix}$$

of $X^{(j)}$, and therefore the linear factors

$$x_1 + r_ix_2 + r_{i3k}x_3 + r_{i4k}x_4 + \cdots + r_{ipk}x_p \qquad \begin{pmatrix} i=1, 2, \cdots, t \\ k=1, 2, \cdots, \alpha_i \end{pmatrix}$$

of the form f_{pm} itself. Thus

Theorem 5: *The complete resolution of a p-ary form f_{pm} which is linearly factorable is accomplished by factoring the binary forms which constitute the rational factors of $X^{(j)}$ in* (15) $(j = 1, 2, \cdots, p - 2)$, *viz.*,

$$\frac{\partial^{\alpha_i}\varphi_{0-r_i}}{\partial r_i^{\alpha_i}} x^{\alpha_i} - \alpha_i \frac{\partial^{\alpha_i-1}\varphi_{1-r_i}{}^{(j)}}{\partial r_i^{\alpha_i-1}} x^{\alpha_i-1}y_i + \cdots + (-1)^{\alpha_i}\alpha_i!\varphi_{\alpha_i-r_i}{}^{(j)}y_i^{\alpha_i} \tag{16}$$

$$(i = 1, 2, \cdots, t).$$

One obtains the field of complete reducibility of f_{pm} by adjoining the $(p-2)t+1$ respective fields of $\varphi_{0\,x_1x_2}$ and the forms (16).

§ 4. Further necessary conditions for reducibility in the field $R(1)$.

Theorem 6: *Let f_{3m} be a completely reducible ternary form having integral coefficients, and let the factors of $\varphi_{0\,x_1x_2}$ be $x_1 + r_ix_2$ $(i = 1, 2, \cdots, t)$, where $x_1+r_ix_2$ is of multiplicity α_i, and r_i belongs to the field $R(1)$. Then if for some prime q and some value of i*

$$\frac{1}{\alpha_i!}\frac{\partial^{\alpha_i}\varphi_{0-r_i}}{\partial r_i^{\alpha_i}} \not\equiv 0 \pmod q, \tag{17}$$

$$\frac{1}{(\alpha_i - k)\,!}\frac{\partial^{\alpha_i-k}\varphi_{k-r_i}}{\partial r_i^{\alpha_i-k}} \equiv 0 \pmod q \qquad (k = 1, 2, \cdots, \alpha_i), \tag{18}$$

a necessary condition that f_{3m} be reducible in the absolute field further than is indicated by formula (13) *and theorem* 4 *is that*

$$\varphi_{\alpha_i-r_i} \equiv 0 \pmod{q^2}. \tag{19}$$

A set of t necessary conditions that f_{3m} be completely reducible in the absolute field is given by (19) *when $i = 1, 2, \cdots, t$.*

Referring to the typical factor in (13),

$$
\begin{aligned}
(20)\qquad & \frac{\partial^{\alpha_i}\varphi_{0-r_i}}{\partial r_i^{\alpha_i}} x^{\alpha_i} - \alpha_i \frac{\partial^{\alpha_i-1}\varphi_{1-r_i}}{\partial r_i^{\alpha_i-1}} x^{\alpha_i-1}y + \cdots \\
& + (-1)^k \alpha_i(\alpha_i-1)\cdots(\alpha_i-k+1)\frac{\partial^{\alpha_i-k}\varphi_{k-r_i}}{\partial r_i^{\alpha_i-k}} x^{\alpha_i-k}y_i^k + \cdots \\
& + (-1)^{\alpha_i}\alpha_i\,!\varphi_{\alpha_i-r_i}y_i^{\alpha_i},
\end{aligned}
$$

we see that every term of this binary form will always be divisible by $\alpha_i\,!$. For, the coefficients in $\partial^{\alpha_i-k}\varphi_{k-r_i}/\partial r_i^{\alpha_i-k}$ that result from differentiation are, when detached,

$$
\begin{aligned}
(21)\qquad & [(m-k)(m-k-1)\cdots(m-\alpha_i+1), \cdots, (a_i-k+2)\cdots 4.3, \\
& \qquad (\alpha_i-k+1)\cdots 3.2, (\alpha_i-k)\cdots 2.1].
\end{aligned}
$$

All of these numbers are divisible by $(\alpha_i-k)\,!$, each being the product of α_i-k consecutive integers. Hence the general term in (20) is divisible by $\alpha_i\,!$; and this is true for $k=0, 1, \cdots, \alpha_i$, which proves the statement. Thus the left hand members of (17), (18) are integers. Removing the factor $\alpha_i\,!$ from the binary form (20), theorem (6) follows directly from Eisenstein's theorem (§ 1).

We now consider two special cases of theorem (6):

$$
\begin{aligned}
(22)\qquad & x_1^3+5x_1^2x_2+8x_1x_2^2+4x_2^3+5x_1^2x_3+15x_1x_2x_3+10x_2^2x_3 \\
& \qquad + (5\beta-\beta^2)x_1x_3^2+(5\beta-\beta^2)x_2x_3^2,
\end{aligned}
$$

$$
\begin{aligned}
(23)\qquad & x_1^4+6x_1^3x_2+13x_1^2x_2^2+12x_1x_2^3+4x_2^4 \\
& + x_1^3x_3 - x_1^2x_2x_3 - 10x_1x_2^2x_3 - 8x_2^3x_3 \\
& - 15x_1^2x_3^2 - 48x_1x_2x_3^2 - 27x_2^2x_3^2 \\
& - 9x_1x_3^3 + 27x_2x_3^3 \\
& + 54x_3^4.
\end{aligned}
$$

In (22) $\varphi_{0\,x_1x_2}$ has a double root r but no triple root; in (23) $\varphi_{0\,x_1x_2}$ has two pairs of double roots r_i $(i=1, 2)$. The forms corresponding to (20) are, after the adventitious factor $2\,!$ is removed,

$$(24)\qquad (-3r+5)x^2-(10r-15)xy-[(5\beta-\beta^2)r-(5\beta-\beta^2)]\,y^2,$$

$$
\begin{aligned}
(25)\qquad & (6r_i^2-18r_i+13)x^2+(3r_i^2+2r_i-10)xy_i \\
& \qquad -(15r_i^2-48r_i+27)y_i^2 \quad (i=1, 2).
\end{aligned}
$$

In (24) $r = 2$, and if we take $q = 5$ the conditions of theorem (6) will be satisfied provided $\beta^2 - 5\beta \equiv 0 \pmod 5$. This makes β a multiple of 5 and $\beta^2 - 5\beta \equiv 0 \pmod{5^2}$, so that the necessary condition expressed by the theorem is satisfied by *both terms* of $(5\beta - \beta^2)r - (5\beta - \beta^2)$ becoming divisible by q^2. We note in passing that the roots of (24) are $-\beta$, $-5 + \beta$, the factored form of (22) being

$$(x_1 + 2x_2 + \beta x_3)[x_1 + 2x_2 + (5 - \beta)x_3](x_1 + x_2).$$

In (25), taking $q = 3$, each term of

$$\varphi_{2-r_i} \equiv -15r_i^2 + 48r_i - 27$$

is divisible by q but all are not divisible by q^2. Hence, if the necessary condition $q^{-1}\varphi_{2-r_i} \equiv 0 \pmod q$ is satisfied, it must be satisfied, not termwise as in (24), but *as a congruence in* r_i. When it is solved as a congruence it determines the multiple root r_i of $\varphi_{0-r_i} = 0$. Solving the quadratic congruence

$$5r_i^2 - 16r_i + 9 \equiv 0 \pmod 3 \tag{26}$$

we get in reality the two roots 2, 0; but only $r_1 = 2$ can be considered, for this is the only one of the two which is a double root of $\varphi_{0-r_i} = 0$. With $r_1 = 2$, all of the conditions of the theorem are satisfied. The other double root of $\varphi_{0-r_i} = 0$ is $r_2 = 1$. The reason it does not appear as a root of (26) is that with $r_2 = 1$

$$3r_2^2 + 2r_2 - 10 \not\equiv 0 \pmod 3,$$

and then the hypotheses of the theorem (6) are not all satisfied. We note that (23) is really completely reducible in the absolute field. The roots of (25) are -3, -3 when $r_i = 2$; and 2, 3, when $r_i = 1$. The factored form is thus, by corollary (4),

$$(x_1 + 2x_2 + 3x_3)(x_1 + 2x_2 + 3x_3)(x_1 + x_2 - 2x_3)(x_1 + x_2 - 3x_3).$$

In the case of an f_{3m} completely reducible in field $R(1)$ and belonging to the case of theorem (6), we now see that when $\varphi_{0\,x_1 x_2}$ has multiple roots r_i the necessary condition

$$\varphi_{\alpha_i - r_i} \equiv 0 \pmod{q^2}$$

may be satisfied by every coefficient of $\varphi_{\alpha_i - r_i}$ being divisible by q^2, or it may be satisfied as a congruence of order $m - \alpha_i$ in r_i. We refer to the former case as being satisfied singularly, and in the latter case say the congruence is satisfied non-singularly. We have

Theorem 7: *If the necessary condition* $\varphi_{a_i - r_i} \equiv 0 \pmod{q^2}$ *is satisfied non-singularly the solution of this congruence determines some, and it may be all, of the multiple roots of the equation* $\varphi_{0-r_i} = 0$.

Theorem 8: *If in a completely reducible ternary form* f_{3m} *of even order* $m = 2n$, $\varphi_{0\,x_1/x_2}$ *has exactly n double roots* r_i *belonging to the absolute field, and if conditions* (28)

$$\frac{1}{2\,!}\frac{\partial^2 \varphi_{0-r_i}}{\partial r_i^2} \not\equiv 0 \pmod{q} \qquad (q \text{ any odd prime}), \tag{27}$$

$$\frac{1}{(2-\kappa)\,!}\frac{\partial^{2-\kappa} \varphi_{\kappa-r_i}}{\partial r_i^{2-\kappa}} \equiv 0 \pmod{q} \qquad (\kappa = 1, 2). \tag{28}$$

are both satisfied singularly, then a necessary condition that f_{3m} *be completely reducible in the absolute field is that the congruence condition*

$$\varphi_{2-r_i} \equiv 0 \pmod{q^2}$$

be satisfied singularly.

In the product

$$\begin{aligned} f_{3m} &= \prod_{i=1}^{m} (x_1 + r_i x_2 + s_i x_3) \\ &= x_1^m + \Sigma r_1 x_1^{m-1} x_2 + \Sigma r_1 r_2 x_1^{m-2} x_2^2 + \cdots \\ &\quad + \Sigma s_1 x_1^{m-1} x_3 + \Sigma s_1 r_2 x_1^{m-2} x_2 x_3 + \Sigma s_1 r_2 r_3 x_1^{m-3} x_2^2 x_3 + \cdots \\ &\quad + \Sigma s_1 s_2 x_1^{m-2} x_3^2 + \Sigma s_1 s_2 r_3 x_1^{m-3} x_2 x_3^2 + \Sigma s_1 s_2 r_3 r_4 x_1^{m-4} x_2^2 x_3^2 + \cdots \\ &\quad + \cdots\cdots\cdots\cdots\cdots\cdots \\ &= x_2^m \varphi_{0\,x_1/x_2} + x_2^{m-1} x_3 \varphi_{1\,x_1/x_2} + x_2^{m-2} x_3^2 \varphi_{2\,x_1/x_2} + \cdots \end{aligned}$$

we have

$$r_{2\mu-1} = r_{2\mu} = \xi_\mu \qquad (\mu = 1, 2, \cdots, n).$$

Hence condition (27) may be written

$$(\xi_j - \xi_1)^2(\xi_j - \xi_2)^2 \cdots (\xi_j - \xi_{j-1})^2(\xi_j - \xi_{j+1})^2 \cdots (\xi_j - \xi_n)^2 \not\equiv 0 \pmod{q} \tag{29}$$

where $r_i = \xi_j$.

Now we can write*

$$\begin{aligned} x_2^{m-1} x_3 \varphi_{1\,x_1/x_2} = \sum_{k=0}^{m-1} \sum_{\mu=1}^{n} \{ s_{2\mu-1} [r_{2\mu} \Sigma r_1 r_2 \cdots r_{2\mu-2} r_{2\mu+1} \cdots r_{k+1} \\ + \Sigma r_1 r_2 \cdots r_{2\mu-2} r_{2\mu+1} \cdots r_{k+2}] + s_{2\mu} [r_{2\mu-1} \Sigma r_1 r_2 \cdots r_{2\mu-2} r_{2\mu+1} \cdots r_{k+1} \\ + \Sigma r_1 r_2 \cdots r_{2\mu-2} r_{2\mu+1} \cdots r_{k+2}] \} x_1^{m-k-1} x_2^k x_3. \end{aligned}$$

* This summation must be *defined* for $k = 0$, and 1. We of course let it stand for the first and second terms of $x_2^{m-1} x_3 \varphi_1 x_1/x_2$ respectively when $k = 0, 1$, i. e., $r_{\mu 0} = 1$, etc.

Evidently, since $r_{2\mu-1}=r_{2\mu}$ the coefficients of $s_{2\mu-1}$, $s_{2\mu}$ are equal in each term of (28), $k=1$. Let these be $\tau_{\mu k}$ $(k=0,1,2,\cdots,m-2)$, and we have

$$\begin{aligned}(30)\qquad \tau_{1k}(s_1+s_2)+\tau_{2k}(s_3+s_4)+\tau_{3k}(s_5+s_6)+\cdots+\tau_{\mu k}(s_{2\mu-1}+s_{2\mu})\\ +\cdots+\tau_{nk}(s_{m-1}+s_m)=\nu_k q.\end{aligned}$$

Solving a properly selected * set of n of these linear equations we get

$$s_{2\mu-1}+s_{2\mu}=\lambda_\mu q \qquad (\mu=1,2,\cdots,n),$$

where λ_μ is an integer if f_{3m} is completely reducible in the field $R(1)$. Hence we have

$$(31)\qquad s_{2\mu-1}\equiv-\sigma_\mu,\quad s_{2\mu}\equiv\sigma_\mu \pmod q.$$

Using (31) in connection with the coefficients of $x_2^{m-2}x_3^2\varphi_{2\,x_1x_2}$ successively we obtain

$$\sigma_1^2+\sigma_2^2+\sigma_3^2+\cdots+\sigma_n^2\equiv0 \pmod q$$
$$\xi_1\sigma_1^2+\xi_2\sigma_2^2+\xi_3\sigma_3^2+\cdots+\xi_n\sigma_n^2\equiv0 \pmod q$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

and by induction

$$(32)\qquad \xi_1^k\sigma_1^2+\xi_2^k\sigma_2^2+\xi_3^k\sigma_3^2+\cdots+\xi_n^k\sigma_n^2\equiv0 \pmod q \qquad (k=0,\ 1,\ 2,\ \cdots,\ n-1,\ \cdots).$$

By (29) none of the ξ_j are congruent to each other. Hence the determinant of the congruences

$$(33)\qquad \left.\begin{aligned}&\sigma_1^2+\sigma_2^2+\sigma_3^2+\cdots+\sigma_n^2\equiv0\\ &\xi_1\sigma_1^2+\xi_2\sigma_2^2+\xi_3\sigma_3^2+\cdots+\xi_n\sigma_n^2\equiv0\\ &\xi_1^2\sigma_1^2+\xi_2^2\sigma_2^2+\xi_3^2\sigma_3^2+\cdots+\xi_n^2\sigma_n^2\equiv0\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &\xi_1^{n-1}\sigma_1^2+\xi_2^{n-1}\sigma_2^2+\xi_3^{n-1}\sigma_3^2+\cdots+\xi_n^{n-1}\sigma_n^2\equiv0\end{aligned}\right\}\pmod q$$

.

* If q happens to be a prime which divides some of the numerical factors of the coefficients in (28), $k=1$, obtained by differentiation, the τ_{ik} $(i=1,2,\cdots,n)$ in such a coefficient are all divisible by q and the corresponding equation (30) must be avoided in selecting n linear equations. Evidently the number of numbers

$$1,\ 2,\ 3,\ \cdots,\ m-2,\ m-1$$

divisible by q will be a maximum, when $q=3$ and $m-1\equiv0 \pmod 3$. This maximum is $\frac{1}{3}(m-1)$. The least number of admissible equations (30), from which the $n=\frac{1}{2}m$ must be selected is thus $\frac{2}{3}(m-1)$, and $\frac{2}{3}(m-1)\geqq\frac{1}{2}m$ when $m\geqq4$. Thus we always have n equations to solve for the $(s_{2\mu-1}+s_{2\mu})$. It is interesting to note that the case $m=2$, to which the analysis above does not apply is not an exception to the theorem; which then reduces to Eisenstein's theorem for a quadratic equation.

is incongruent to zero, so that we have at once

$$\sigma_1 \equiv \sigma_2 \equiv \sigma_3 \equiv \cdots \equiv \sigma_n \equiv 0 \pmod{q}.$$

Thus every s_i is divisible by q, and every coefficient of φ_{2-r_i} by q^2, which was to be proved.

Corollary 5: *Further necessary conditions are that the congruences*

$$\varphi_{\nu-r_i} \equiv 0 \ (mod\ q^\nu) \qquad (\nu = 3,\ 4,\ \cdots,\ m)$$

be all satisfied singularly.

A theorem analogous to theorem 8 holds when f_{3m} is of odd order $m = 2n + 1$ and φ_{0-r} has one simple root and n double roots.

PHILADELPHIA, PA.

A DETERMINANT FORMULA FOR THE NUMBER OF WAYS OF COLORING A MAP.

By George D. Birkhoff.

Suppose that a finite set of two-dimensional regions making up a simply or multiply connected closed surface are given, so that these form a *map* M. Each of these regions may be taken to be limited by closed curves, formed by a finite number of continuous *boundary lines* which the region has in common with other regions. The ends of these lines, at which three or more regions meet, are called *vertices* of the map. A *coloring* of the map consists in attributing to each region a color different from that of any region having in common with it a boundary line, but not necessarily different from that of a region meeting it at a vertex.

The following fact will first be proved: *The number of ways of coloring the given map M in λ colors $(\lambda = 1, 2, \cdots)$ is given by a polynomial $P(\lambda)$ of degree n, where n is the number of regions of the map M.* In fact let $m_i (i = 1, 2, \cdots, n)$ be the number of ways of coloring the map by using exactly i colors *when mere permutations of the colors are disregarded.* With this definition it is clear that

$$m_i \cdot \lambda \cdot (\lambda - 1) \cdots (\lambda - i + 1)$$

represents the number of ways of coloring the given map in exactly i of the λ colors *counting two colorings as distinct when they are obtained by a permutation one from the other*; for, of the i colors used, the first may be chosen in λ ways, the second in $\lambda - 1$ ways, and so on. If λ is less than i the above term reduces to zero.

But the total number of ways of coloring the given map in λ colors is the sum of the number of ways of coloring it with $1, 2, \cdots, n$ of these colors, since no more than n colors can be used. Accordingly the total number of ways is represented by

$$P(\lambda) = m_1\lambda + m_2\lambda(\lambda - 1) + \cdots + m_n\lambda(\lambda - 1) \cdots (\lambda - n + 1)$$

for all values of λ. It is clear that in general $m_1 = 0$ inasmuch as for $n > 1$ no map can be colored in a single color, and that $m_n = 1$ since there is only one way of coloring M in n colors if permutation of the colors be disregarded.

In order to proceed to the effective determination of $P(\lambda)$ we consider the total number $\mu - 1$ of ways of forming from the map $M^{(1)} = M$ *submaps* $M^{(2)}, \cdots, M^{(k)}$ of $n - 1$ regions, $M^{(k+1)}, \cdots, M^{(l)}$ of $n - 2$ regions, and so on to $M^{(\mu)}$ of one region, by successive coalescence of regions adjacent along a boundary line. Such a coalescence may be indicated by the removal of all the common boundary lines of the two regions which coalesce. The maps $M^{(2)}, \cdots, M^{(k)}$ are obtained by one such step, the maps $M^{(k+1)}, \cdots, M^{(l)}$ by two such steps, and so on.

At this point we introduce the symbol (i, k) to denote the number of ways of breaking down the map M in n regions to a submap of i regions by k simple or multiple coalescences, i. e., by picking out maps $M, M^{(a_1)}, \cdots, M^{(a_k)}$, each but the first being a submap of the preceding one, and the last one having i regions. It is apparent that we have $(i, k) = 0$ for $k > n - i$, and that $(i, n - i)$ represents the number of ways of making $n - i$ successive simple coalescences. By definition we take $(n, 0) = 1$ and $(i, 0) = 0$ for $i < n$.

Let now one of the λ colors be placed at random on each of the regions of any map $M^{(i)}$ of the μ maps above defined. Each one of these arrangements will color *one and one only* of the maps $M^{(i)}$ and its submaps $M^{(i_1)}$, $M^{(i_2)}, \cdots$, namely that one obtained by a coalescence of all adjacent regions which receive the same colors. In consequence if we let $\sigma_1, \sigma_2, \cdots, \sigma_\mu$ denote the number of ways of coloring $M^{(1)}, M^{(2)}, \cdots, M^{(\mu)}$ respectively in λ colors, we will have

$$\lambda^{n_i} = \sigma_i + \sigma_{i_1} + \cdots \qquad (i = 1, 2, \cdots, \mu),$$

in which the symbol n_i denotes the number of regions in $M^{(i)}$, and λ^{n_i} is then the total number of ways of giving one of the λ colors to each region; on the right appear the numbers $\sigma_i, \sigma_{i_1}, \sigma_{i_2}, \cdots$ corresponding to $M^{(i)}$ and its submaps $M^{(i_1)}, M^{(i_2)}, \cdots$. Let ϵ_{ij} for $i \neq j$ be 1 or 0 according as $M^{(i)}$ does or does not contain $M^{(j)}$ as a submap, and let ϵ_{ii} be 1; we may write the above equations in the form

$$\lambda^{n_i} = \sum_{j=1}^{\mu} \epsilon_{ij}\sigma_j \qquad (i = 1, 2, \cdots, \mu).$$

This set of μ equations is linear in the μ quantities $\sigma_1, \cdots, \sigma_\mu$. If we observe that because of the arrangement of the submaps of M according to a decreasing number of regions we have $i_1 > i, i_2 > i, \cdots$ in the above equations, it becomes clear that $\epsilon_{ij} = 0$ for $i > j$. Thus the determinant of this system of equations is 1 and therefore solving for $\sigma_1 = P(\lambda)$ we obtain

$$P(\lambda) = \begin{vmatrix} \lambda^{n_1}, & \epsilon_{12}, & \cdots, & \epsilon_{1n} \\ \lambda^{n_2}, & \epsilon_{22}, & \cdots, & \epsilon_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda^{n_\mu}, & \epsilon_{\mu 2}, & \cdots, & \epsilon_{\mu n} \end{vmatrix}$$

a determinant formula for the number of ways of coloring the map in λ colors.

This determinant is of the order μ, approximately of the magnitude $n\,!$. Furthermore it might be proved that the quantities ϵ_{ij} determine the constitution of the map, so that if the determinant were written in full the structure of the complete map might be deduced from it. These two facts show the complicated character of the determinant.

The evaluation of this determinant may be carried out in terms of the symbols (i, k) previously introduced. To this end let us consider a typical term

$$\pm \lambda^{n_j} \epsilon_{\alpha 2} \epsilon_{\beta 3} \cdots \epsilon_{\kappa \mu}$$

where the $-$ or $+$ sign is taken according as $j, \alpha, \beta, \cdots, \kappa$ gives an odd or even permutation of $1, 2, \cdots \mu$, i. e., according as the substitution

$$\begin{pmatrix} 1 & 2 & \cdots & \mu \\ j & \alpha & \cdots & \kappa \end{pmatrix}$$

is the product of an odd or an even number of transpositions.

Any such term either reduces to zero or to $\pm \lambda^{n_j}$. We shall consider how terms not zero may arise.

If the above substitution is not the identical substitution it may always be decomposed into a product of cyclic substitutions $(\rho_1, \rho_2, \cdots, \rho_k)$ composed of k elements and changing ρ_1 to ρ_2, ρ_2 to ρ_3, $\cdots$, ρ_k to ρ_1. Such a cyclic substitution must contain the element 1; else there arises in the term a product of factors

$$\epsilon_{ab},\ \epsilon_{bc},\ \cdots,\ \epsilon_{la}$$

which is zero necessarily since we cannot have simultaneously $a < b$, $b < c, \cdots, l < a$. *It follows that the substitution degenerates into a single cyclic substitution at most, containing the element* 1.

The corresponding term is thus of the form

$$\pm \lambda^{n_j} \epsilon_{ja} \epsilon_{ab} \cdots \epsilon_{l1} \epsilon_{mm} \epsilon_{pp} \cdots,$$

where the $+$ or $-$ sign is to be taken according as the cyclic substitution $(1, j, a, \cdots, l)$ contains an odd or an even number of elements. Conversely to every product of this sort which is not zero we have a single term not zero of the determinant.

Suppose now that we attempt to obtain the sum of all the terms of this kind for a given $n_j = i$ and a given number of elements $k + 1$ of this cyclic substitution. If none of the factors ϵ_{ja}, ϵ_{ab}, $\cdots$ are to be zero the

map $M^{(j)}$ contains $M^{(a)}$ as a submap, the map $M^{(a)}$ contains $M^{(b)}$ as a submap, and so on. Thus we obtain a sequence of $k+1$ maps M, $M^{(j)}$, $M^{(a)}, \cdots, M^{(l)}$, each after the first a submap of the preceding one. Consequently there is one and only one such term corresponding to each way of breaking down M in k steps to some submap of i regions, and each such term has the same sign as $(-1)^k$.

The stated terms therefore are $(-1)^k(i, k)\lambda^i$ in value and the final formula for the number of ways of coloring the given map in λ colors is

$$P(\lambda) = \sum_{i=1}^{n} \lambda^i \sum_{k=0}^{n-i} (-1)^k(i, k).$$

The term in λ^n, corresponding to the identical substitution, has the proper coefficient unity according to our previous convention by which the symbol $(n, 0)$ has the value 1.

As a first example of the formula we take the very simple case of a map of three regions which are adjacent each to each. We will have

$$(2,\ 1) = 3, \quad (1,\ 1) = 1, \quad (1,\ 2) = 3,$$

and

$$P(\lambda) = (3,\ 0)\lambda^3 - (2,\ 1)\lambda^2 + [-(1,\ 1) + (1,\ 2)]\lambda = \lambda(\lambda - 1)(\lambda - 2).$$

The validity of this formula may be verified at once by noticing that we can color any one of these three regions in λ colors, a second region in the $\lambda - 1$ remaining colors, and the third region in the $\lambda - 2$ colors left after the first two regions are colored.

As a second example we take the case of a map in five regions formed by a ring of three regions bounding an interior and exterior region. In this case the symbols (i, k) which enter have the values

$$\begin{array}{llll} (4, 1) = 9, & (3, 1) = 22, & (3, 2) = 51; & \\ (2, 1) = 14, & (2, 2) = 125, & (2, 3) = 150; & \\ (1, 1) = 1, & (1, 2) = 45, & (1, 3) = 176, & (1, 4) = 150 \end{array}$$

so that

$$P(\lambda) = \lambda^5 - 9\lambda^4 + 29\lambda^3 - 39\lambda^2 + 18\lambda = \lambda(\lambda - 1)\ (\lambda - 2)\ (\lambda - 3)^2$$

In this case also the validity of the formula may be at once verified, for the three regions of the ring must be in three distinct colors, while the interior and exterior regions may be in any fourth color different from these three colors.

Even in this second case the value of the symbols (i, k) is not immediately obtained; and if we have a somewhat more complicated map, for example the map formed by twelve five-sided regions on the sphere, a considerable computation would be necessary to determine $P(\lambda)$ directly from the formula, or from the map itself.

PRINCETON UNIVERSITY,
May 4, 1912.

TWO THEOREMS ON CONICS.

By S. Lefschetz.

1. This paper contains two unrelated theorems on conics, of which the first is virtually contained in a proposition given by Poncelet,* while the second is the equivalent for conics of a well-known theorem on rational quartics, and so far as we know is new under this form. To both propositions we apply considerations of number of projective conditions satisfied by a given configuration.

2. The first theorem is the following: *When two complete quadrilaterals circumscribed to a conic have a common diagonal, their vertices, not situated on the latter, are on another conic.*

Let OAB be the triangle of reference, and

$$S = x^2 - yz = 0,$$

the equation of a conic S tangent to OA and OB in A and B. The two tangents drawn from the point a $(0, 1, \lambda)$ are simultaneously represented by:

$$4\lambda(x^2 - yz) + (\lambda y + z)^2 = 0,$$

which can be written:

$$4\lambda x^2 - (\lambda y - z)^2 = 0.$$

By changing λ into μ, we obtain the tangents drawn from the point b $(0, 1, \mu)$, so that

$$4\lambda x^2 - (\lambda y - z)^2 + \kappa[4\mu x^2 - (\mu y - z)^2] = 0 \tag{1}$$

represents a conic going through the intersections of the two pairs of tangents. Similarly a conic going through the points of intersection of the tangent from the point a' $(0, 1, \lambda')$ with the tangents from the point b' $(0, 1, \mu')$ will be represented by

$$4\lambda' x^2 - (\lambda' y - z)^2 + \kappa'[4\mu' x^2 - (\mu' y - z)^2] = 0 \tag{2}$$

and we must show that for proper values of κ and κ', the equations (1) and (2) will represent the same curve.

Such values will satisfy the system:

$$\frac{\lambda + \kappa\mu}{\lambda' + \kappa'\mu'} = \frac{\lambda^2 + \kappa\mu^2}{\lambda'^2 + \kappa'\mu'^2} = \frac{1 + \kappa}{1 + \kappa'},$$

* Traité des propriétés projectives des figures, T. 1, § 568. This was pointed out to the writer by E. Fabry.

which effectively can be solved for κ and κ'. Thus we get

$$\kappa = -\frac{(\lambda - \lambda')(\lambda - \mu')}{(\mu - \lambda')(\mu - \mu')}$$

and substituting in (1) we obtain:

$$2(\lambda\mu - \lambda'\mu')(2x^2 - yz) + [\lambda\mu(\lambda' + \mu') - \lambda'\mu'(\lambda + \mu)]y^2 + [(\lambda + \mu) - (\lambda' + \mu')]z^2 = 0$$

as the equation of the conic Σ circumscribed to the two quadrilaterals circumscribed to S. Now the pairs of points (a, b) and (a', b') determine on AB an involution, the parameters of which are defined by:

$$\alpha\lambda\mu + \beta(\lambda + \mu) + \gamma = 0,$$
$$\alpha\lambda'\mu' + \beta(\lambda' + \mu') + \gamma = 0,$$

which gives for the equation of Σ:

$$\Sigma = 2\beta(2x^2 - yz) - \gamma y^2 - \alpha z^2 = 0,$$

which shows that Σ depends solely on the involution on AB.

3. Let C and D be the double points of the involution just considered on AB, they clearly determine entirely the position of Σ with respect to S, and if E is one of the points of intersection of OC and the conic S, then the whole system is projectively determined by the four points O, A, B, E, and the cross ratio $(ABCD)$. Hence projectively there are ∞^1 systems such as that formed by S and Σ. Since there are projectively ∞^2 systems of two conics, there must be one relation between the invariants of the system (S, Σ). Effectively, if D is the discriminant of

$$S + \rho\Sigma = 0$$

we have:

$$D = \rho^3\Delta_1 + \theta_1\rho^2 + \theta_2\rho + \Delta_2$$
$$= -[\tfrac{1}{4}\rho^3 + 2\beta\rho^2 + (5\beta^2 - \alpha\gamma)\rho + 4\beta(\beta^2 - \alpha\gamma)],$$

Δ_1, Δ_2 being the discriminants of S and Σ, and θ_1, θ_2 their relative invariants. Identifying we get:

$$\Delta_1 = -\tfrac{1}{4}, \quad \theta_1 = -2\beta, \quad \theta_2 = -(5\beta^2 - \alpha\gamma),$$
$$\Delta_2 = -4\beta(\beta^2 - \alpha\gamma).$$

As these four form a complete invariant system of two conics, and only three of them contain β and $(\alpha\gamma)$, by eliminating these two parameters we

can obtain one and only one relation. We have:

$$4\theta_2\Delta_1 = 5\beta^2 - \alpha\gamma = \tfrac{5}{4}\theta_1^2 - \alpha\gamma$$

$$\frac{8\Delta_2\Delta_1^2}{\theta_1} = \beta^2 - \alpha\gamma = \theta_1^2 - \alpha\gamma.$$

Hence the relation desired is *

$$\theta_1^3 - 4\Delta_1\theta_1\theta_2 + 8\Delta_2\Delta_1^2 = 0,$$

expressing the condition that there exist a quadrilateral circumscribed to S and inscribed to Γ. If we put $(ABCD) = \sigma$, then $\sigma = m_1/m_2$, m_1 and m_2 being the roots of

$$-\alpha m^2 + 2\beta m + \gamma = 0.$$

Hence σ is given by the equation

$$\frac{\alpha\gamma}{4\beta^2}(\sigma + 1)^2 - \sigma = 0$$

and if

$$J = \frac{\Delta_1\Delta_2}{\theta_1\theta_2},$$

is the absolute invariant of the system, then

$$\sigma^2 + 2\frac{(14J - 3)}{10J - 1}\sigma + 1 = 0.$$

4. The second proposition can be stated thus: *If the tangents at three of the common points of a conic and a triangle go through the opposite vertices, then the triangle is self polar with respect to the conic.*†

There are two possible cases according as the three points are, or are not on different sides.

In the first case let ABC be a triangle, a_1, a_2, b_1, b_2, c_1, c_2 its intersections with a conic S, the tangents at a_1, b_1, c_1 going respectively through the vertices A, B, C. Let D be the intersection of Bb_1 and Cc_1, and E and F, the points where b_1c_1 meets BC and AD respectively. We have $(EFc_1b_1) = -1$ and hence AD is polar of E with respect to S. Hence the polar of A goes through E, and as it goes also through a_1, it must be Ea_1, or BC. Similarly for B and C, which proves the proposition in this case.

In the second case suppose that S is tangent to Bb_1, Bb_2, Cc_1. Then B

* The expression given is an invariant since it is homogeneous and of the 9th degree in the coefficients of two general conics.

† Transforming quadratically with respect to the triangle we obtain this: *If a rational quartic has more than three flecnodes, it has necessarily three biflecnodes.* For proofs see: Basset, Treatise on cubic and quartic curves, p. 111; J. E. Rowe, "A complete system of invariants of the rational quartic," Transactions of the American Mathematical Society, Vol. 12, p. 309.

is pole of AC and therefore the polar of c going through B and c_1 must be Bc_1 or AB. Since C and B are poles of the opposite sides the proposition is true in this case too.

5. Another proof will be given here. The system formed by a triangle and an arbitrary conic depends upon three absolute invariants, for it is a degenerate quintic with nine double points, and since the most general curve of the fifth order depends upon 12 absolute invariants, between which each new double point establishes a relation, the truth of our statement follows. Otherwise also: It is easily shown that the system is projectively completely defined by the three cross ratios (ABc_1c_2), (BCa_1a_2), (CAb_1b_2). That there is no relation between them is seen thus: To any four points a_1, a_2, b_1, b_2 taken arbitrarily on the proper sides, there corresponds a pencil of conics which determines on AB an involution, a pair of which will be defined in one of two ways by (ABc_1c_2), which is thus shown to be arbitrary when the other two cross-ratios are given.

Now if we impose upon the system the conditions that one or two of the tangents to the conic from the vertices meet it on the sides, they alone will do so, and the conditions are independent. For there is a conic tangent to Aa_1 in a_1, going through B and tangent to an arbitrary line at its intersection with AB, and for this conic Aa_1 is the only tangent from the vertices meeting it on the sides. Next, among the conics tangent to Aa_1 in a_1, and to Bb_1 and b_1, there is one tangent to AB at a certain point K, with respect to which b_1K and a_1K will be the polars of A and B, and therefore again Aa_1, and Bb_1 are the only tangents from the vertices which will meet this conic on the sides. The same is evident for conics tangent to Aa_1 and Aa_2 in a_1 and a_2, since for them AB and AC are arbitrary.

If now we impose upon a conic the conditions that three of the tangents at the intersections with the sides of the triangle go through the opposite vertices, these conditions being projective and independent, will determine the cross ratios on the sides. Hence any two systems satisfying these conditions for which the cross ratios are definite, will be projectively equivalent. But such a system is presented by a conic and any one of its self-polar triangles, into which the system considered can therefore be projected —and this proves the proposition.

LINCOLN, NEBR.,
November 28, 1911.

A NEW TYPE OF SOLUTION OF LAPLACE'S EQUATION.

By H. Bateman.

1. In the work of classifying solutions* of the potential equation $\nabla^2 V = 0$, the problem of constructing solutions involving arbitrary functions deserves considerable attention.

There are really two problems of this kind, but they are intimately related with one another. In the first problem the aim is to find potential functions of the form †

$$V = f(X, Y), \tag{1}$$

in which X and Y are certain functions of x, y, z and f satisfies a partial differential equation in the two variables X, Y.

This problem has been solved by Levi Civita and generalized by Amaldi‡ who considers solutions of the form

$$V = uf(X, Y), \tag{2}$$

where f satisfies a partial differential equation. This generalized problem can be discussed by means of a relation of the form§

$$(dx^2 + dy^2 + dz^2)(p^2 + q^2 + r^2) - (pdx + qdy + rdz)^2 = AdX^2 + 2HdXdY + BdY^2$$

in which A, H, B are functions of X and Y.

In the second problem the aim is to find solutions of the form

$$V = uf(\theta), \tag{3}$$

where f is an arbitrary function with a finite second derivative and u, θ are certain functions of x, y, z. It is known that when a solution of this form exists, the parameter θ must satisfy the differential equation of the characteristics

$$\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 + \left(\frac{\partial\theta}{\partial z}\right)^2 = 0 \tag{4}$$

and this fact has been used by Forsyth|| to obtain a class of solutions of the form (3). His result is as follows:

* This work has been commenced by Levi Civita, Torino Memoirs, vol. XLIX (1899), pp. 105–152. A different classification is given in Kelvin-Tait's Natural Philosophy.

† Volterra has called these functions binary potentials. "Sopra alcuni problemi della teoria del potenziale," Ann. Sc. Normale di Pisa, 1883.

‡ Rend. Palermo, vol. 16 (1902), pp. 1–45.

§ Cambr. Phil. Trans., vol. 21 (1910), pp. 257–280.

|| Mess. Math., vol. XXVII (1898), pp. 99–118; Phil. Trans. A., vol. CXCI (1898). The first part of the theorem is due to Jacobi, Werke, Bd. 2, p. 208.

Let p, q, r be three functions of θ such that

$$p^2 + q^2 + r^2 = 0$$

and let θ be defined in terms of x, y, z by means of the equation

$$\theta = px + qy + rz;$$

then $V = f(\theta)$ is a solution of $\nabla^2 V = 0$ and a second solution is given by

$$V = \frac{g(\theta)}{1 - xp'(\theta) - yq'(\theta) - zr'(\theta)}.$$

Some developments of this theorem have been given by Burnside* and Bromwich.† The latter obtains a class of solutions of the form

$$V = Xf(u) + Yf'(u).$$

Forsyth has obtained some analogous theorems for the equation of wave motion, showing in particular that if α, β, γ, σ are four functions of θ satisfying the equation

$$\sigma^2 = \alpha^2 + \beta^2 + \gamma^2$$

and θ be defined in terms of x, y, z, t by the equation

$$F(\theta) = x\alpha(\theta) + y\beta(\theta) + z\gamma(\theta) - t\sigma(\theta),$$

then $V = f(\theta)$ is a solution of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial t^2} \tag{5}$$

and

$$\frac{X(\theta)}{F'(\theta) - x\alpha'(\theta) - y\beta'(\theta) - z\gamma'(\theta) + t\sigma'(\theta)}$$

is another solution of this equation.

A solution analogous to this is well known in the theory of electrons, as it occurs in the specification of an electromagnetic field with a simple singularity moving through the æther.‡ It is obtained as follows: Let θ be defined by the equation

$$[x - \xi(\theta)]^2 + [y - \eta(\theta)]^2 + [z - \zeta(\theta)]^2 = [t - \tau(\theta)]^2; \tag{6}$$

then if

$$M = \xi'(\theta)[x - \xi(\theta)] + \eta'(\theta)[y - \eta(\theta)] + \zeta'(\theta)[z - \zeta(\theta)] - \tau'(\theta)[t - \tau(\theta)]$$

the function $V = f(\theta)/M$ satisfies (5) and θ is a solution of the equation

$$\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 + \left(\frac{\partial\theta}{\partial z}\right)^2 = \left(\frac{\partial\theta}{\partial t}\right)^2. \tag{7}$$

* Mess. Math., vol. XXVII (1898), pp. 138–146. See also Schottky, Berl. Ber. (1909.)

† Proc. London Math. Soc., Ser. 1, vol. 30 (1899).

‡ A. Liénard, L'éclairage électrique (1898), pp. 5, 53, 106; E. Wiechert, Archives néerlandaises (2), 5 (1900), p. 54; A. W. Conway, Proc. London Math. Soc., Ser. 2, vol. 1, p. 154, 1903; H. Bateman, Ibid., 1911.

The corresponding solution of Laplace's equation does not seem to have been given hitherto, probably because it is of a slightly different form. If we define θ in terms of x, y, z by the equation*

$$[x - \xi(\theta)]^2 + [y - \eta(\theta)]^2 + [z - \zeta(\theta)]^2 = 0 \tag{8}$$

and write

$$M = \xi'(\theta)[x - \xi(\theta)] + \eta'(\theta)[y - \eta(\theta)] + \zeta'(\theta)[z - \zeta(\theta)]$$

it appears that $1/M^k$ is not generally a solution of Laplace's equation for any value of k. To obtain a solution in this case we proceed as follows: Let l, m, n be three functions of θ satisfying the equations

$$\left.\begin{aligned} l\xi'(\theta) + m\eta'(\theta) + n\zeta'(\theta) &= 0, \\ l^2 + m^2 + n^2 &= 0, \end{aligned}\right\} \tag{9}$$

and let w be defined by the equation

$$w = l(x - \xi) + m(y - \eta) + n(z - \zeta). \tag{10}$$

Differentiating we find that

$$\frac{\partial w}{\partial x} = l + K\frac{\partial \theta}{\partial x}, \quad \frac{\partial w}{\partial y} = m + K\frac{\partial \theta}{\partial y}, \quad \frac{\partial w}{\partial z} = m + K\frac{\partial \theta}{\partial z}$$

where

$$K = l'(x - \xi) + m'(y - \eta) + \eta'(z - \zeta).$$

These equations combined with (4) and (9) give

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 = 2K\left[l\frac{\partial \theta}{\partial x} + m\frac{\partial \theta}{\partial y} + n\frac{\partial \theta}{\delta z}\right]. \tag{11}$$

The second derivatives of w are given by formulæ of the type

$$\frac{\partial^2 w}{\partial x^2} = 2l'\frac{\partial \theta}{\partial x} + N\left(\frac{\partial \theta}{\partial x}\right)^2 + K\frac{\partial^2 \theta}{\partial x^2},$$

where the coefficient N is the same in each. Adding these we get

$$\nabla^2 w = 2\left[l'\frac{\partial \theta}{\partial x} + m'\frac{\partial \theta}{\partial y} + n'\frac{\partial \theta}{\partial z}\right] + K\nabla^2\theta. \tag{12}$$

Now

$$x - \xi = [\xi'(x - \xi) + \eta'(y - \eta) + \zeta'(z - \zeta)]\frac{\partial \theta}{\partial x} = M\frac{\partial \theta}{\partial x}.$$

Differentiating we find that

$$1 - \xi'\frac{\partial \theta}{\partial x} = M\frac{\partial^2 \theta}{\partial x^2} + \left(\xi' + L\frac{\partial \theta}{\partial x}\right)\frac{\partial \theta}{\partial x}$$

* This makes θ a solution of equation (4).

and two similar equations with the same coefficient L. Adding these we get

$$3 - 2\left(\xi' \frac{\delta\theta}{\partial x} + \eta' \frac{\partial\theta}{\partial y} + \zeta' \frac{\partial\theta}{\partial z} \right) = M\nabla^2\theta.$$

But the equations

$$x - \xi = M\frac{\partial\theta}{\partial x}$$

give

$$\xi' \frac{\partial\theta}{\partial x} + \eta' \frac{\partial\theta}{\partial y} + \zeta' \frac{\partial\theta}{\partial z} = 1;$$

therefore

$$M\nabla^2\theta = 1.$$

Equations (11) and (12) may now be written in the form

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 = \frac{2Kw}{M}, \quad \nabla^2 w = \frac{3K}{M},$$

and consequently

$$2w\nabla^2 w - 3\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] = 0.$$

This means that the function

$$u = \frac{1}{\sqrt{w}}$$

is a solution of $\nabla^2 u = 0$.

It should be noticed that equations (9) give two sets of values for the *ratios* of the quantities l, m, n. If one appropriate set of values of l, m, n is known, another set can be obtained by multiplying each quantity by $f(\theta)$; the corresponding function w becomes multiplied by $f(\theta)$ and we infer that the function

$$V = \frac{g(\theta)}{\sqrt{w}}, \quad g(\theta) = \frac{1}{\sqrt{f(\theta)}}$$

is a solution of $\nabla^2 V = 0$.

If X, Y, Z are interpreted as current coördinates of a point, the plane whose equation is

$$l[X - \xi] + m[Y - \eta] + n[Z - \zeta] = 0$$

is a common tangent plane of the curve

$$X = \xi(\lambda), \quad Y = \eta(\lambda), \quad Z = \zeta(\lambda),$$

and the circle at infinity. It is easy to see that the function $u = 1/\sqrt{w}$ becomes infinite at points of this curve; hence we have constructed a potential function whose singularities lie along an arbitrary curve. To illustrate the theorem let us first of all define θ by the equation

$$x^2 + y^2 + (z - \theta)^2 = 0;$$

this gives

$$\theta = z \pm i\rho. \qquad (\rho^2 = x^2 + y^2)$$

The equations for determining l, m, n, are

$$n = 0, \quad l^2 + m^2 = 0$$

hence we may take $w = x \pm iy$ and we find that

$$\frac{1}{\sqrt{x \pm iy}} f(z \pm i\rho)$$

is a solution of Laplace's equation. Putting,

$$x = \rho \cos \phi, \quad y = \rho \sin \phi,$$

we obtain real solutions of the form*

$$\rho^{-\frac{1}{2}} \cos \frac{\phi}{2} [f(z + i\rho) + f(z - i\rho)],$$

$$\rho^{-\frac{1}{2}} \sin \frac{\phi}{2} [f(z + i\rho) + f(z - i\rho)],$$

which are periodic in ϕ with the period 4π. Next, let us define θ by the equation

$$(x - a \cos \theta)^2 + (y - a \sin \theta)^2 + z^2 = 0.$$

The equations for determining l, m, n are

$$l \sin \theta - m \cos \theta = 0, \quad l^2 + m^2 + n^2 = 0.$$

Taking $l = \cos \theta$, $m = \sin \theta$, $n = i$, we have

$$w = \cos \theta(x - a \cos \theta) + \sin \theta\,(y - a \sin \theta) + iz;$$

hence

$$2aw = x^2 + y^2 + (z + ia)^2,$$

and we conclude that the function

$$V = \frac{f(\theta)}{\sqrt{x^2 + y^2 + (z + ia)^2}}$$

is a solution of Laplace's equation.

Putting $x = \rho \cos \phi$, $y = \rho \sin \phi$ and introducing the coördinates σ, ψ defined by the equations

$$\rho = \frac{a \sinh \sigma}{\cosh \sigma - \cos \psi}, \quad z = \frac{a \sin \psi}{\cosh \sigma - \cos \psi}$$

* These are well known.

we find that

$$\frac{1}{[\rho^2 + (z + ia)^2]^{\frac{1}{2}}} = e^{-\frac{i\psi}{2}} \sqrt{\cosh \sigma - \cos \psi},$$

$$\cos(\theta - \phi) = \frac{a^2 + z^2 + \rho^2}{2a\rho} = \coth \sigma, \quad e^{i(\theta-\phi)} = \tanh \frac{\sigma}{2}.$$

Hence the solution

$$\frac{e^{im\theta}}{[\rho^2 + (z + ia)^2]^{\frac{1}{2}}}$$

is transformed into

$$e^{-\frac{i\psi}{2}} \sqrt{\cosh \sigma - \cos \psi} \cdot e^{im\phi} \tanh^m \frac{\sigma}{2},$$

and we have real solutions of the form

$$\sqrt{(\cosh \sigma - \cos \psi)} \cos \frac{\psi}{2} \cdot \cos(m\phi + \epsilon)\left(\tanh \frac{\sigma}{2}\right)^m,$$

$$\sqrt{(\cosh \sigma - \cos \psi)} \sin \frac{\psi}{2} \cdot \cos(m\phi + \epsilon)\left(\tanh \frac{\sigma}{2}\right)^m,$$

It is well known that Laplace's equation is satisfied by a function of the form

$$V = \sqrt{(\cosh \sigma - \cos \psi)} \cos n\psi \cos m\phi\, P_{n-\frac{1}{2}}^m (\cosh \sigma);$$

the solutions just obtained belong to this class, for the differential equation for the associated Legendre function $P_{n-\frac{1}{2}}^m (\cosh \sigma)$ is satisfied by $(\tanh \sigma/2)^m$ when $n = \frac{1}{2}$.

It should be noticed that a solution of the equation

$$\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{\partial \theta}{\partial z}\right)^2 = 0$$

may also be obtained by defining θ and α by means of the equations

$$[x - \xi(\theta, \alpha)]^2 + [y - \eta(\theta, \alpha)]^2 + [z - \zeta(\theta, \alpha)]^2 = 0,$$

$$\frac{\partial \xi}{\partial \alpha}(x - \xi) + \frac{\partial \eta}{\partial \alpha}(y - \eta) + \frac{\partial \zeta}{\partial \alpha}(z - \zeta) = 0.$$

I have not yet succeeded in finding a function u such that $uf(\theta)$ is a solution of Laplace's equation.

Bryn Mawr College,
April, 1912.

INVOLUTORIC CIRCULAR TRANSFORMATIONS AS A PARTICULAR CASE OF THE STEINERIAN TRANSFORMATION AND THEIR INVARIANT NETS OF CUBICS.*

By Arnold Emch.

It is well known that with every Steinerian † or involutoric quadratic transformation in a plane is associated an invariant net of cubics passing through the fundamental quadruple of the transformation and its diagonal points. Conversely, every plane cubic admits of an infinite number of birational transformations in itself, which form a continuous group. In case of a rational cubic (deficiency 0) the group depends upon three parameters. For a cubic of deficiency 1, there exists a mixed one-parameter group consisting of a linear system g_2^1 of Steinerian transformations, and a group of non-central transformations. The transformations of the first kind do not form a group and are involutoric, while those of the second kind form a one-parameter group with three involutoric transformations. The product of two transformations of the first kind is equivalent with a transformation of the second kind.‡

The involutoric circular transformation in a complex plane

$$z' = \frac{az + b}{cz - a}$$

is a particular case of a Steinerian transformation. It is the purpose of this paper to establish this equivalence and the principal properties of the two nets of cubics which remain invariant in the transformations of the first and second kind, respectively.

In order to gain a general standpoint, I shall first briefly discuss the well-known conditions for conformal rational transformations in a plane, and then apply the results to birational quadratic transformations. This will lead immediately to general circular—and in particular to circular transformations in the Steinerian form.§

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* Read before the American Mathematical Society in Chicago, Dec. 29, 1911.

† Steiner, Werke, vol. I, pp. 409–421; Disteli, "Die Metrik der circularen Curven dritter Ordnung," Vierteljahrsschrift Zürich, vol. 37, pp. 255–305.

‡ Weyer, "Über eindeutige Beziehungen auf einer allgemeinen ebenen Curve dritter Ordnung," Wiener Ber., vol. 87, pp. 837–872; Segre, "Remarques sur les transformations uniformes des courbes elliptiques en elles-mêmes," Math. Ann., vol. 27, p. 296.

§ A discussion of the five Newtonian types of plane cubics and their construction by means of the Steinerian transformation was given by the author in the Univ. of Colorado Studies, vol. 1, pp. 275–284.

1. General Rational Algebraic Transformations.

Let R_1, R_2, R_3, R_4 designate polynomials in x and y of any degree, then

$$x' = \frac{R_1}{R_3}, \qquad y' = \frac{R_2}{R_4} \tag{1}$$

represents a general rational algebraic transformation of the xy-plane into the $x'y'$-plane. It is furthermore assumed that the fractions R_1/R_3 and R_2/R_4 are irreducible. To every point (x, y) corresponds uniformly a point (x', y'). Conversely, to a point (x', y') correspond the points of intersection of the curves $x' R_3 - R_1 = 0$ and $y' R_4 - R_2 = 0$. Thus, in case of a general quadratic transformation, there is a $(1, 4)$ correspondence between the points of the two planes. Applying the conditions for conformity

$$\frac{\partial x'}{\partial x} = \frac{\partial y'}{\partial y}, \qquad \frac{\partial x'}{\partial y} = -\frac{\partial y'}{\partial x} \tag{2}$$

to (1), we get

$$\frac{\dfrac{\partial R_1}{\partial x} R_3 - \dfrac{\partial R_3}{\partial x} R_1}{R_3^2} = \frac{\dfrac{\partial R_2}{\partial y} R_4 - \dfrac{\partial R_4}{\partial y} R_2}{R_4^2}, \tag{3}$$

$$\frac{\dfrac{\partial R_1}{\partial y} R_3 - \dfrac{\partial R_3}{\partial y} R_1}{R_3^2} = \frac{\dfrac{\partial R_2}{\partial x} R_4 - \dfrac{\partial R_4}{\partial x} R_2}{R_4^2}. \tag{4}$$

(3) and (4) must be satisfied for all possible pairs of values of x and y. But two rational fractions in x and y which are equal for all pairs of values of x and y have identical numerators and denominators (except as to a factor of proportionality which we may assume as cancelled). Hence $R_4 = R_3$. The other possibility $R_4 = -R_3$ may be disregarded, since it may be obtained from the first merely by a reflection on the x-axis. Thus putting in (3) and (4) $R_4 = R_3$, from these equations follows

$$\left(\frac{\partial R_1}{\partial x} R_3 - \frac{\partial R_3}{\partial x} R_1\right)\left(\frac{\partial R_2}{\partial x} R_3 - \frac{\partial R_3}{\partial x} R_2\right) + \left(\frac{\partial R_1}{\partial y} R_3 - \frac{\partial R_3}{\partial y} R_1\right)\left(\frac{\partial R_2}{\partial y} R_3 - \frac{\partial R_3}{\partial y} R_2\right) = 0,$$

or, dividing through by R_3, and replacing R_1/R_3 by x' and R_2/R_3 by y', and developing:

$$x'y'\left[\left(\frac{\partial R_3}{\partial x}\right)^2 + \left(\frac{\partial R_3}{\partial y}\right)^2\right] - x'\left(\frac{\partial R_3}{\partial x}\cdot\frac{\partial R_2}{\partial x} + \frac{\partial R_3}{\partial y}\cdot\frac{\partial R_2}{\partial y}\right) - y'\left(\frac{\partial R_1}{\partial x}\cdot\frac{\partial R_3}{\partial x} + \frac{\partial R_1}{\partial y}\cdot\frac{\partial R_3}{\partial y}\right) + \left(\frac{\partial R_1}{\partial x}\cdot\frac{\partial R_2}{\partial x} + \frac{\partial R_1}{\partial y}\cdot\frac{\partial R_2}{\partial y}\right) = 0. \tag{5}$$

This, as can easily be established, is the condition that all curves of the two pencils $x'R_3 - R_1 = 0$ and $y'R_3 - R_2 = 0$ are orthogonal to each other, and in case of conformity is a well-known fact. From (5) we conclude that also *the curves $R_1 = 0$, $R_2 = 0$ and $R_3 = 0$ are orthogonal among themselves.* To the infinite corresponds the curve $R_3 = 0$, which according to (5), when $x' = \infty$, $y' = \infty$, must satisfy the differential equation

$$(6) \qquad \left(\frac{\partial R_3}{\partial x}\right)^2 + \left(\frac{\partial R_3}{\partial y}\right)^2 = 0.$$

The general solution of this equation is of the form

$$(7) \qquad [\phi(x, y)]^2 + [\psi(x, y)]^2 = 0,$$

in which both $\phi\,(x, y)$ and $\psi\,(x, y)$ satisfy Laplace's differential equation.* In case of a rational transformation ϕ and ψ are polynomials in x and y. The real part of the curve $R_3 = 0$ consists therefore only of a limited number of points, namely the real points of intersection of the two curves

$$\phi(x, y) = 0 \qquad \text{and} \qquad \psi(x, y) = 0.$$

As $R_1 = 0$ and $R_2 = 0$ are orthogonal to $R_3 = 0$, they necessarily pass through these points of intersection. By means of the foregoing properties it is now easy to establish conformal quadratic transformations.

2. Conformal Quadratic Transformations.

In this case the pencils $x'R_3 - R_1 = 0$ and $y'R_3 - R_2 = 0$ must form two orthogonal pencils of conics. This is possible only when the conics are circles, so that the two pencils are conjugate. But as R_3 is common to both pencils, the real part of the circle $R_3 = 0$ must necessarily degenerate into a point. The two pencils have therefore a common point and any curve of the first pencil cuts every curve of the second pencil in one point only. From this it follows that the transformation is birational and of the form†

$$(8) \qquad \begin{cases} x' = \dfrac{A_2B_3 - A_3B_2}{A_1B_2 - A_2B_1}, \\[2ex] y' = \dfrac{A_3B_1 - A_1B_3}{A_1B_2 - A_2B_1}, \end{cases}$$

where the A's and B's represent arbitrary linear expressions in x and y. According to (2) and (7) $R_3 = A_1B_2 - A_2B_1$ has the form

$$(c_1x - c_2y + d_1)^2 + (c_1y + c_2x + d_2)^2,$$

* There is of course: $\dfrac{\partial\phi}{\partial x} = \dfrac{\partial\psi}{\partial y}$, $\dfrac{\partial\phi}{\partial y} = -\dfrac{\partial\psi}{\partial x}$.

† K. Doehlemann: Geometrische Transformationen, vol. II, pp. 56–78.

hence

$$-B_1 = A_2 = c_1x - c_2y + d_1,$$

and

$$A_1 = B_2 = c_1y + c_2x + d_2.$$

Substituting these expressions in (8) and replacing A_3 and B_3 also by linear expressions the transformation assumes the form

$$x' = \frac{(\alpha_1x + \alpha_2y + b_1)(c_1x - c_2y + d_1) + (a_1y + a_2x + b_2)(c_1y + c_2x + d_2)}{(c_1x - c_2y + d_1)^2 + (c_1y + c_2x + d_2)^2},$$

$$y' = \frac{(a_1y + a_2x + b_2)(c_1x - c_2y + d_1) - (\alpha_1x + \alpha_2y + b_1)(c_1y + c_2x + d_2)}{(c_1x_1 - c_2y + d_1)^2 + (c_1y + c_2x + d_2)^2}.$$

It will be noticed that so far the coefficients α_1, α_2, b_1, $-a_1$, $-a_2$, $-b_2$ are entirely arbitrary and independent of c_1, c_2, d_2. The numerators of these expressions however must also represent circles, which requires the equality of the coefficients of x^2 and y^2 in each case and the vanishing of the coefficients of xy. This leads to the two conditions

$$\alpha_1c_1 + \alpha_2c_2 = a_1c_1 - a_2c_2,$$

$$\alpha_1c_2 - \alpha_2c_1 = a_1c_2 + a_2c_1.$$

Solving for α_1 and α_2, we find $\alpha_1 = a_1$, $\alpha_2 = -a_2$. Writing $p_1 = a_1x - a_2y + b_1$, $q_1 = c_1x - c_2y + d_1$, $p_2 = a_1y + a_2x + b_2$, $q_2 = c_1y + c_2x + d_2$ the equations of the circles $R_1 = 0$ and $R_2 = 0$ may be written in the form

$$(9) \qquad R_1 \equiv p_1q_1 + p_2q_2 = 0, \qquad R_2 \equiv p_2q_1 - p_1q_2 = 0.$$

From this is seen that both circles pass through the points of intersection of the rays $p_1 = 0$ and $p_2 = 0$, and $q_1 = 0$ and $q_2 = 0$. The first circle (R_1) passes also through the intersection of $p_1 = 0$ and $q_2 = 0$, and also of $q_1 = 0$ and $p_2 = 0$. The second circle (R_2) passes through the vertices of the pairs $p_1 = 0$, $q_1 = 0$ and $p_2 = 0$, $q_2 = 0$, Fig. 1. The real portion of $R_3 = 0$ is clearly the point of intersection of $q_1 = 0$ and $q_2 = 0$. From the equations of the rays p_1, p_2, q_1, q_2 it is seen that $p_1 \perp p_2$ and $q_1 \perp q_2$. (In Fig. 1 the letters contained in the equations are used to designate the corresponding rays and circles.) Hence, p_1 and q_1 are altitudes in the triangle XYZ, and also ZM, cutting XY in Z_1, is an altitude and $\triangle MY_1Z \sim \triangle XY_1Y$. These two triangles are brought into a homologous position by turning them through an angle of 90°, and as the circles R_1 and R_2 are also homologously related to the triangles, it follows that in their original position they are orthogonal. This as we have seen in the general case, is a necessary condition. The orthogonality of R_1 and R_2 might of course also be proved analytically without difficulty. Writing

the transformations in the form

$$(10) \qquad x' = \frac{p_1q_1 + p_2q_2}{q_1^2 + q_2^2}, \qquad y' = \frac{p_2q_1 - p_1q_2}{q_1^2 + q_2^2},$$

and making use of the identities

$$\frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial y} = 0, \qquad \frac{\partial p_1}{\partial y} + \frac{\partial p_2}{\partial x} = 0, \qquad \frac{\partial q_1}{\partial x} - \frac{\partial q_2}{\partial y} = 0, \qquad \frac{\partial q_1}{\partial y} + \frac{\partial q_2}{\partial x} = 0,$$

it is easily verified that (10) satisfies (2). *The transformation is therefore conformal.* Substituting in the equation $ux' + vy' + w = 0$ for x' and y' their expressions as given by (10), the equation

$$u(p_1q_1 + p_2q_2) + v(p_2q_1 - p_1q_2) + w(q_1^2 + q_2^2) = 0$$

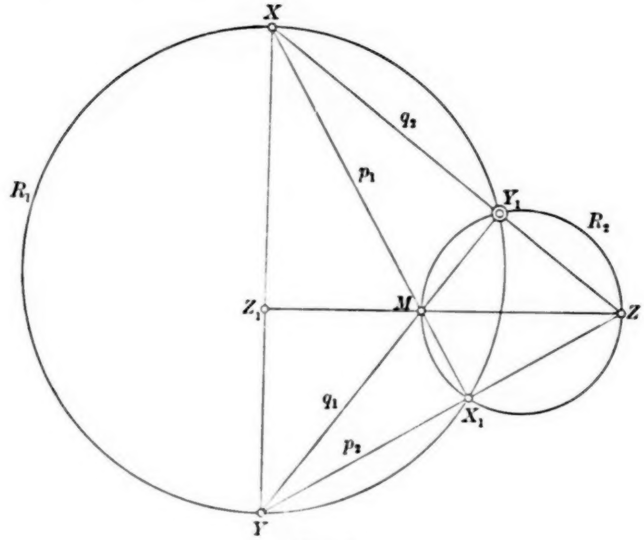

FIG. 1.

results. In each parenthesis the coefficients of x^2 and y^2 are equal, so that this equation represents a circle passing through Y_1, ($R_3 = 0$), the point corresponding to the infinite point of the straight line in the $x'y'$-plane. Consider now a point P' in the $x'y'$-plane and the pencil of rays through it. This pencil is transformed into the pencil of circles through Y_1 and P (corresponding to P') in the xy-plane. Any circle with P' as a center is transformed into a circle of the conjugate pencil in the xy-plane. Thus every circle in the $x'y'$-plane is transformed into a circle of the xy-plane. Conversely, every circle in the xy-plane may be considered as a circle of a pencil which is conjugate to a pencil having Y_1 as one of the base points. The two pencils are transformed into a pencil of rays and the corresponding systems

of concentric circles. Every circle of the xy-plane is therefore transformed into a circle of the $x'y'$-plane. The transformation is therefore circular. By this the theorem is established:

All conformal quadratic transformations are circular transformations.

It remains to be shown that the transformation as represented by (14) may be expressed as a transformation between complex variables. For this purpose we have from (10)

$$x' + iy' = \frac{p_1q_1 - i \cdot ip_2q_2 + ip_2q_1 - ip_1q_2}{(q_1 + iq_2)(q_1 - iq_2)},$$

or

$$x' + iy' = \frac{(p_1 + ip_2)(q_1 - iq_2)}{(q_1 + iq_2)(q_1 - iq_2)},$$

or

$$x' + iy' = \frac{p_1 + ip_2}{q_1 + iq_2} = \frac{a_1x - a_2y + b_1 + i(a_1y + a_2x + b_2)}{c_1x - c_2y + d_1 + i(c_1y + c_2x + d_2)}.$$

Replacing $-a_2y$ by $ia_2 \cdot iy$ and $-c_2y$ by $ic_2 \cdot iy$, this, by factoring, becomes

$$x' + iy' = \frac{(a_1 + ia_2)(x + iy) + b_1 + ib_2}{(c_1 + ic_2)(x + iy) + d_1 + id_2},$$

or, in the usual form

(11) $$z' = \frac{az + b}{cz + d}.$$

3. Involutoric Circular Transformation.

If in (11) $d = -a$, or $d_1 = -a_1$, $d_2 = -a_2$, the transformation becomes involutoric:

(12) $$z' = \frac{az + b}{cz - a},$$

or in Cartesian coördinates

(13) $$\begin{cases} x' = \dfrac{(a_1x - a_2y + b_1)(c_1x - c_2y - a_1) + (a_1y + a_2x + b_2)(c_1y + c_2x - a_2)}{(c_1x - c_2y - a_1)^2 + (c_1y + c_2x - a_2)^2} \\[2ex] y' = \dfrac{(a_1y + a_2x + b_2)(c_1x - c_2y - a_1) - (a_1x - a_2y + b_1)(c_1y + c_2x - a_2)}{(c_1x - c_2y - a_1)^2 + (c_1y + c_2x - a_2)^2}. \end{cases}$$

The general involutoric quadratic (Steinerian) transformation of a plane in itself may be defined in the following manner:* Let

$$P_1 \equiv \alpha_1x^2 + 2\beta_1xy + \gamma_1y^2 + 2\delta_1x + 2\epsilon_1y + \varphi_1 = 0,$$
$$P_2 \equiv \alpha_2x^2 + 2\beta_2xy + \gamma_2y^2 + 2\delta_2x + 2\epsilon_2y + \varphi_2 = 0$$

* Steiner, Disteli, loc. cit. R. Sturm: Die Lehre von den geometr. Verwandtschaften, vol. II, pp. 72–95; A. Emch: Introduction to Projective Geometry, pp. 185–216.

represent two distinct conics. Then the polars of a point (x', y') with respect to the pencil

$$P_1 + \lambda P_2 = 0$$

are concurrent at a point (x, y). Conversely all polars of the point (x, y) with respect to the pencil pass through the point (x', y'). Thus, to every point (x, y) corresponds a point (x', y') and conversely. The transformation established in this manner is involutoric and quadratic. It is evident that any two conics of the pencil, consequently also two of the degenerate conics may serve as a base of the transformation. Designate the quadrangle through which the pencil passes by $A_1A_2A_3A_4$ and its diagonal points by B_1, B_2, B_3. To find P' when P is given, join P to B_1, B_2, B_3, Fig. 2, and construct the fourth harmonic rays to PB_1, PB_2, PB_3 respectively

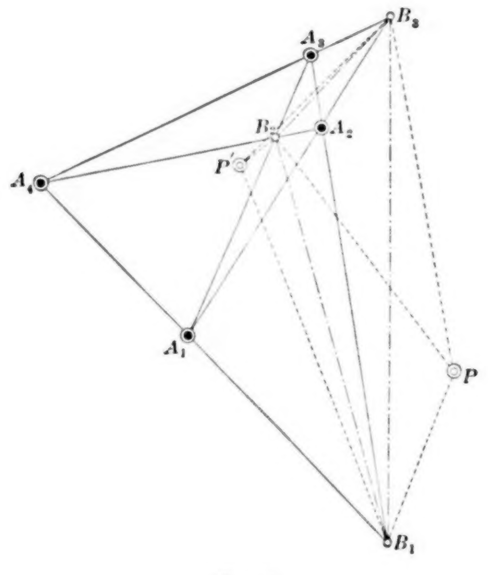

FIG. 2.

with respect to the corresponding pairs of lines (degenerate conics) through B_1, B_2, B_3. The three harmonic rays intersect each other at P'. To the points B correspond all points of their opposite sides of the triangle $B_1B_2B_3$. The points A_1, A_2, A_3, A_4 are invariant. To a point on any line joining any two A's, say A_1A_3, corresponds the fourth harmonic point to the pair A_1A_3 on the same line. All other points are in uniform correspondence. To the straight lines of the plane corresponds the net of conics through $B_1B_2B_3$. To the line at infinity corresponds a conic through these points which bisects the sides of the quadrilateral. This conic is a circle if two of the degenerate conics consist of rectangular pairs of lines. In this case the conics of the

pencil are all equilateral hyperbolas. Conversely, two equilateral hyperbolas determine a pencil, whose degenerate conics consist of rectangular pairs of lines and all of whose conics are equilateral hyperbolas.

The analytic form of the general involutoric quadratic transformation is

$$(14)\quad \begin{cases} x' = \dfrac{(\beta_1 x + \gamma_1 y + \epsilon_1)(\delta_2 x + \epsilon_2 y + \varphi_2) - (\beta_2 x + \gamma_2 y + \epsilon_2)(\delta_1 x + \epsilon_1 y + \varphi_1)}{(\alpha_1 x + \beta_1 y + \delta_1)(\beta_2 x + \gamma_2 y + \epsilon_2) - (\alpha_2 x + \beta_2 y + \delta_2)(\beta_1 x + \gamma_1 y + \epsilon_1)}, \\[2ex] y' = \dfrac{(\alpha_2 x + \beta_2 y + \delta_2)(\delta_1 x + \epsilon_1 y + \varphi_1) - (\alpha_1 x + \beta_1 y + \delta_1)(\delta_2 x + \epsilon_2 y + \varphi_2)}{(\alpha_1 x + \beta_1 y + \delta_1)(\beta_2 x + \gamma_2 y + \epsilon_2) - (\alpha_2 x + \beta_2 y + \delta_2)(\beta_1 x + \gamma_1 y + \epsilon_1)}. \end{cases}$$

In order that this be at the same time a circular transformation, we must dispose of the coefficients in such a manner that (14) becomes identical with (13). This is the case, if we put

$$\begin{array}{llll} \alpha_1 = c_2, & \varphi_2 = b_1, & \alpha_2 = -c_1, & \varphi_1 = -b_2 \\ \beta_1 = c_1, & \gamma_2 = c_1, & \beta_2 = c_2, & \gamma_1 = -c_2 \\ \delta = d_2 = -a_2, & \epsilon_2 = d_2 = -a_2, & \delta_2 = -d_1 = a_1, & \epsilon_1 = d_1 = -a_1. \end{array}$$

The conics which determine the pencil are now

$$(15)\quad \begin{cases} P_1 \equiv c_2 x^2 + 2c_1 xy - c_2 y^2 - 2a_2 x - 2a_1 y - b_2 = 0, \\ P_2 \equiv -c_1 x^2 + 2c_2 xy + c_1 y^2 + 2a_1 x - 2a_2 y + b_1 = 0. \end{cases}$$

Eliminating from these equations in turn xy, and x^2 and y^2, the two new conics belonging to the same pencil arise:

$$(16)\quad \begin{cases} Q_1 \equiv x^2 - y^2 - 2\dfrac{a_1c_1 + a_2c_2}{c_1^2 + c_2^2}x + 2\dfrac{a_2c_1 - a_1c_2}{c_1^2 + c_2^2}y - \dfrac{b_1c_1 + b_2c_2}{c_1^2 + c_2^2} = 0, \\[2ex] Q_2 \equiv 2xy - 2\dfrac{a_2c_1 - a_1c_2}{c_1^2 + c_2^2}x - 2\dfrac{a_1c_1 + a_2c_2}{c_1^2 + c_2^2}y + \dfrac{b_1c_2 - b_2c_1}{c_1^2 + c_2^2} = 0. \end{cases}$$

They clearly represent two equilateral hyperbolas. All conics of the pencil are therefore equilateral hyperbolas and the degenerate conics are three pairs of rectangular lines with the points B_1, B_2, B_3 as vertices. To the line at infinity corresponds a circle through these three points, whose equation is

$$(17)\quad R_3 \equiv (c_1x - c_2y - a_1)^2 + (c_1y + c_2x - a_2)^2 = 0.$$

The real portion of this circle is the point Y_1 with the coördinates

$$(18)\quad s = \frac{a_1c_1 + a_2c_2}{c_1^2 + c_2^2}, \qquad t = \frac{a_2c_1 - a_1c_2}{c_1^2 + c_2^2},$$

which according to (16) is apparently the common center of the two hyperbolas $Q_1 = 0$, $Q_2 = 0$. As the circles corresponding to the straight lines

of the plane all pass through B_1, B_2, B_3, it follows that these points coincide with the point Y_1 ($R_3 = 0$) and the circular points at infinity. The quadrangle $A_1A_2A_3A_4$ has therefore only two real points, while the remaining two are conjugate imaginary. Writing $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$, and forming $Q_1 + iQ_2 = 0$, and reducing we find

$$cz^2 - 2az - b = 0, \tag{19}$$

which is recognized as the equation for the double points of the transformation. The real points of intersection of the two hyperbolas coincide with these double-points and are therefore given by

$$A_1 \equiv \frac{a + \sqrt{a^2 + bc}}{c}, \qquad A_2 \equiv \frac{a - \sqrt{a^2 + bc}}{c}. \tag{20}$$

The middle-point B_3 between A_1 and A_2 is determined by a/c and in $z' = (az + b)/(cz - a)$ clearly corresponds to the point at infinity $z = \infty$. B_3 is the center of the involution and its coördinates, resulting by separating real and imaginary of a/c, are identical with s and t. Thus B_3 coincides with Y_1. The pencil of circles through A_1 and A_2 and its conjugate orthogonal pencil are invariant in the transformation. The two imaginary points of intersection of all circles of the conjugate pencil are identical with the imaginary pair A_3A_4 of the quadruple determined by the hyperbolas.* From the equations of these we see that any two equilateral orthogonal hyperbolas determine such a quadrangle. Hence, the theorem may be stated:

The Steinerian transformation (involutoric quadratic transformation) based on a pencil of conics determined by two equilateral orthogonal hyperbolas is identical with the involutoric circular transformation (involution) having the real pair of the quadruple of the pencil as double points.

4. The Invariant Cubics.

Designating by P and P' two corresponding points of a Steinerian transformation based on a quadruple $A_1A_2A_3A_4$ with the diagonal points $B_1B_2B_3$, every cubic through these nine points is left invariant by the transformation. With every point P in a general position is associated such a cubic, so that there is a net of cubics through the seven points (A) and (B), in which every cubic is invariant. Geometrically nothing is lost in generality if we assume the center of involution B_3 as the origin and the

* Darboux in "Sur une classe remarquable de courbes et de surfaces algébriques," pp. 61–66, calls A_3A_4 "points associés" of A_1A_2, and conversely.—Study in "Vorlesungen über ausgewählte Gegenstände der Geometrie. Erstes Heft, pp. 8–19, calls A_1A_2 the "first picture" (erstes Bild) of A_3A_4.

line joining A_1A_2 as the real axis. We may also assume $B_3A_1 = 1$, so that the circular transformation assumes the simple form

$$z' = \frac{1}{z}. \tag{21}$$

The equations of the hyperbolas determining the pencil of conics through the fundamental quadruple are now

$$Q_1 \equiv x^2 - y^2 - 1 = 0, \qquad Q_2 \equiv 2xy = 0. \tag{22}$$

The polars of a point $P(x_1, y_1)$ with respect to these conics have the equations

$$q_1 \equiv x_1x - y_1y - 1 = 0, \qquad q_2 \equiv x_1y + y_1x = 0.$$

The two pencils $Q_1 + \lambda Q_2 = 0$ and $q_1 + \lambda q_2$ are projective and produce the circular cubic $Q_1q_2 - Q_2q_1 = 0$, or explicitly:

$$(x^2 + y^2)(y_1x - x_1y) + 2xy - y_1x - x_1y = 0. \tag{23}$$

This cubic is also generated by all pairs of corresponding points on the rays of the pencil through (x_1, y_1). Hence the theorem:

The locus of all pairs of corresponding points of a circular transformation on the rays of a pencil in a general position is a circular cubic.

The tangents from

$$P'\left(x_1' = \frac{x_1}{x_1^2 + y_1^2}, \qquad y_1' = \frac{-y_1}{x_1^2 + y_1^2}\right)$$

to the cubic pass through B_3, B_1, B_2, i. e., they have these as points of tangency. As B_1 and B_2 are the circular points at infinity, P' is the real focus or center of the cubic. The fourth (real) tangent passes through P as the point of tangency. The direction of the real asymptoteis determined by the slope y_1/x_1 or the ray through P and B_3. The equation of the asymptote is therefore $y = (y_1/x_1)x + k$, in which k is easily determined as $k = 2y_1/(x_1^2 + y_1^2)$. The asymptote cuts the cubic in the principal point H with the abscissa $h = x_1(k^3 + k)/2(k - k^2y_1 - y_1)$ and the ordinate $j = (y_1/x_1)\,h + k$.

The properties of the inversion $z' = 1/z$ lead to the following results: Every pair of corresponding or inverse points on the cubic, like A and A' on a ray through P is concyclic with the double points A_1 and A_2, Fig. 3; in fact $(AA'A_1A_2) = -1$. The same pair of points also lies on a circle of the orthogonal conjugate pencil of circles. The same is true for any other pair of inverse points on the cubic. Hence the theorem:

The circular cubic (elliptic serpentine) may be produced by the pencil of circles through A_1A_2 and a projective conjugate pencil.

Designating the first pencil of circles by (S_1), the second by (S_2) and the corresponding pencil of rays through P by (T), the theorem follows:

The same cubic may be produced by either pencil (S_1) or (S_2) and the corresponding projective pencil (T).

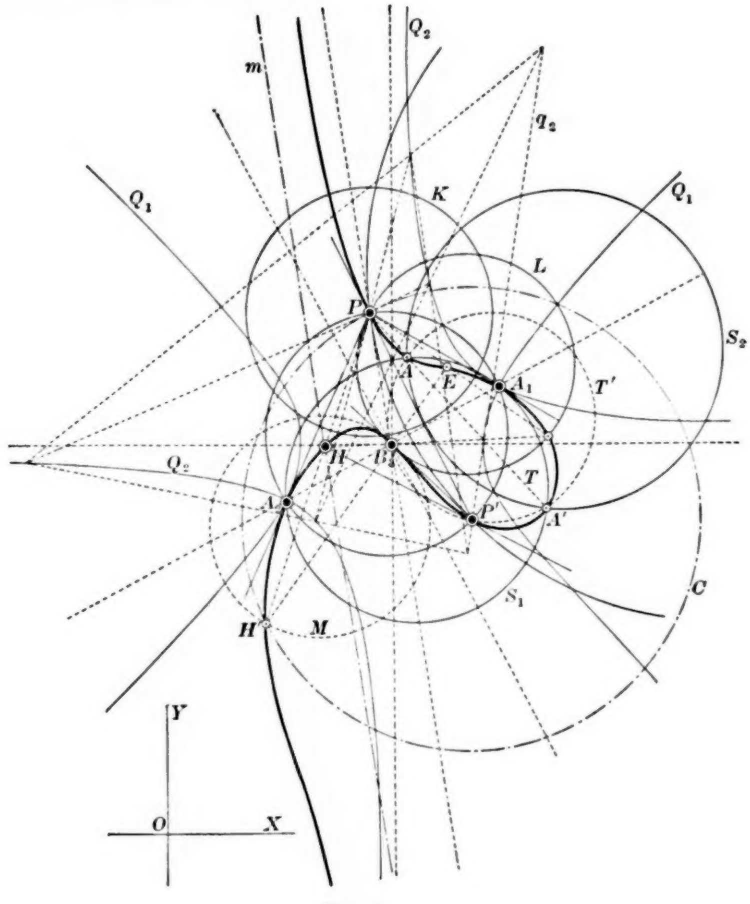

FIG. 3.

If S_1, S_2, T is a corresponding set in this projectivity, the tangents from P to S_1 and S_2 are equal, since $PA \cdot PA'$ equals the squares of these tangents. S_1 and S_2 are therefore orthogonal to a circle K with P as a center. Conversely any pair of (S_1) and (S_2) simultaneously orthogonal to a circle of the concentric pencil around P cuts out two inverse points on the cubic. Hence the theorem:

If for every circle of a concentric pencil an orthogonal pair (orthogonal to the given circle) from two conjugate pencils of circles is determined, then the locus of the points of intersection of all such pairs is a circular cubic.

By means of this theorem it is easy to construct any number of points of the cubic. To get such a pair of points draw any circle S_1 of the first pencil; next draw the circle K with P as a center and orthogonal to S_1; then draw a circle of (S_2) (center on A_1A_2 produced) orthogonal to S_1 and K. The intersections A and A' of S_1 and S_2 are two required points.

By the inversion $z' = 1/z$ the pencils (S_1) and (S_2) are transformed into themselves, while the pencil of rays (T) is transformed into the pencil of circles through B_3 and P', (T'), and there is clearly

$$(T') \barwedge (S_1) \barwedge (S_2).$$

The cubic is therefore also produced by the following projectivities: (T') and (T), (T') and (S_1), (T') and (S_2). It is known that every conic of a pencil of conics through four points of a cubic cuts the cubic in two other points and that all rays joining two such points concur in a point of the cubic. Hence, every tangent circle to the cubic at B_3 cuts the cubic in two points and the ray joining them passes through a fixed point E of the cubic. In the figure this point was determined by means of the tangent-circle L. Conversely, every ray through E cuts the cubic in two points which with B_3 determine a tangent-circle at B_3. Hence, connecting E with B_3 and producing, the cubic is cut in a point H', so that three points of the cubic and on the corresponding tangent circle are now assembled at B_3. This circle, M with B_3H' as a diameter, is therefore osculating the cubic at B_3, and its inverse in the real asymptote m of the cubic. The inverse H of H', through which m passes is called the principal point of the cubic.

The concentric pencil of circles around P' touches the cubic at the circular points. The rays connecting every pair of points of intersection of circles of this pencil with the cubic concur in a point of the cubic. In the limiting case of an infinitely large circle (with P' as a center), the ray joining the two points of intersection is the asymptote. Hence, H is the point of concurrence of those rays, and the well-known theorems follow:

Every ray through the principal point H of the circular cubic cuts the latter in two points which are equidistant from the center (real focus) P'.

The circular cubic is the product of a pencil of rays and a projective pencil of concentric circles.

An interesting particular case is obtained when the point P, Fig. 3, is assumed infinitely distant. In this case the circular points at infinity are conjugate with respect to the cubic and the points P' and H coincide with B_3, Fig. 4. The curve is symmetric with respect to B_3 and the tangents at A_1 and A_2 are parallel to the asymptote m. The construction of the curve itself is remarkably simple: Draw any line perpendicular to m and cutting the coördinate-axes at σ_1 and σ_2; $\sigma_1\sigma_2$ corresponds to one of the circles

K of the concentric pencil around P in the general case. With σ_1 as a center and $\sigma_1 A_2$ as a radius draw a circle S_1; from σ_2 draw tangents to S_1; the points of tangency of these, A and A', are two corresponding points of the cubic and are on a ray $T \parallel m$. A and A' are, of course, also situated on the circle S_2 of the conjugate pencil to (S_1), and on a circle T' tangent to the cubic at B_3. B_3 is a point of inflexion of the cubic. Designating the slope of m by μ, the equation of the cubic, according to (23), becomes

$$(x^2 + y^2)(\mu x - y) - \mu x - y = 0. \tag{24}$$

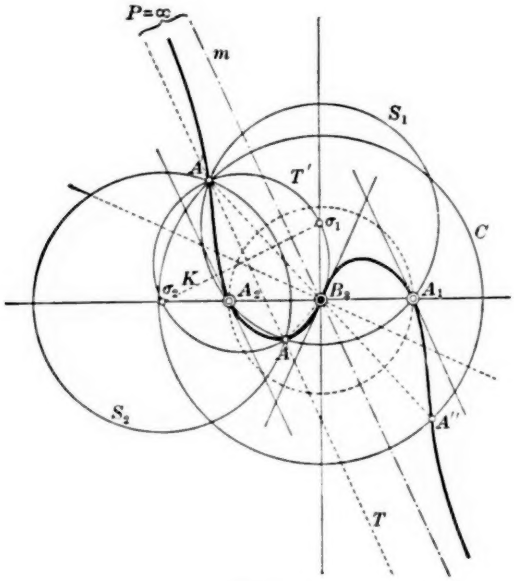

FIG. 4.

5. The Invariant Nets.

It has been seen that with every point P in a Steinerian transformation is associated an invariant cubic. If the transformation is at the same time circular, the cubic also will be circular and of the elliptic serpentine type. In an involutoric circular transformation, there exists therefore a net of such cubics, of which every individual remains invariant. *The net is produced by all the projectivities that may be established between the two conjugate pencils of circles having the double points of the transformation as a base.* Two circles chosen at random from the two pencils determine a transformation, which clearly is possible in a doubly infinite number of ways. The equation of the net belonging to the transformation $z' = 1/z$ is

$$(x^2 + y^2)(y_1 x - x_1 y) + 2xy - y_1 x - x_1 y = 0. \tag{25}$$

These transformations of the cubic in itself, or central correspondences, are of the first kind. There are ∞^4 such transformations and consequently that many nets and ∞^6 circular elliptic serpentines admitting of such transformations.

The second kind of involutoric circular transformations of a cubic in itself, the noncentral correspondences on the cubic, are obtained in the following manner. In Fig. 5 let O be the origin; on the real axis

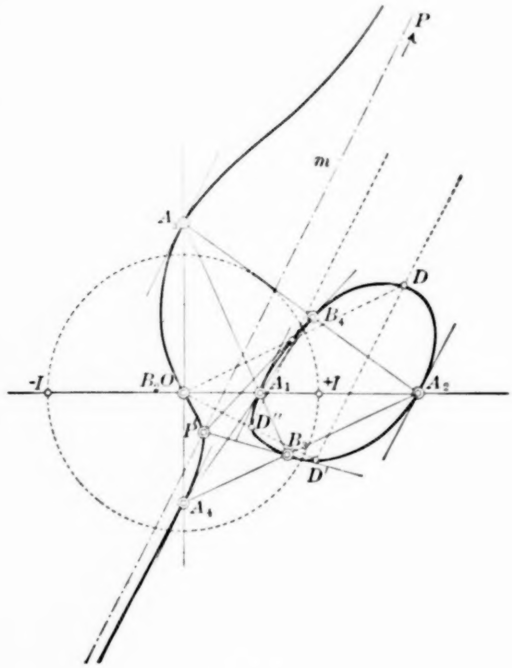

FIG. 5.

$OA_1 \cdot OA_2 = +1$; and on the imaginary axis $OA_3 \cdot OA_4 = -1$. Draw the diagonal points $B_2(\equiv 0)$, B_3, B_4, and construct a cubic associated with an infinitely distant point P, in the Steinerian transformation belonging to the quadruple $A_1A_2A_3A_4$. Such a cubic passes through the circular points at infinity, hence through nine points (the two circular points, the A's and B's) which as a whole remain invariant in the inversion $z' = 1/z$. Designating by $P = 0$ and $Q = 0$ the equations of any two conics through the A's and by $p = 0$ and $q = 0$ the polars of $P\infty$ with respect to $P = 0$ and $Q = 0$, the equation of the cubic will be $Pq - Qp = 0$. If the coördinates of A_i are x_i, y_i, we may assume for P and Q the expressions

$$P \equiv xy \quad \text{and} \quad Q \equiv (xx_1y_3 + y - y_3)(x - yx_1y_3 - x_1),$$

so that the polars of $P\infty$ with respect to P and Q are represented by

$$p \equiv y + mx,$$

and

$$q \equiv [x_1y_3 + \frac{m}{2}(1 - x_1^2y_3^2)]x + [\tfrac{1}{2}(1 - x_1^2y_3^2) - mx_1y_3]y$$
$$- \tfrac{1}{2}[x_1^2y_3 + y_3 + m(x_1 - x_1y_3^2)].$$

The equation of the cubic in its explicit form is now

$$(26) \quad (y - mx)(x^2 + y^2) + \frac{m(1 - x_1^2)}{x_1}x^2 + \frac{1 - y_3^2}{y_3}y^2 - mx - ey = 0.$$

For every set of values x_1, y_3, m an invariant cubic is obtained. With every inversion $z' = 1/z$ a triply-infinite system of invariant cubics may thus be established. For all these cubics the transformation is non-central, or of the second kind, and is one of the three which are involutoric and of the second kind for all circular cubics through the fundamental quadruple $A_1A_2A_3A_4$ and its diagonal points. The other two have their centers at B_3 and B_4. Geometrically the relation between the transformations J and E of the first and second kind is as stated at the beginning. By a J with P as a center a point D is transformed to D'; by a J with O as a center D' is transformed to D''. By an E with O as a center, D is transformed to D''. Thus, the product of two J's is an E.

As there are ∞^4 circular involutoric transformations in a plane, there are as many systems of invariant cubics, or ∞^7 circular cubics bearing non central involutoric correspondences. As the general equation of a circular cubic depends on 7 essential constants it follows, that by means of Steinerian transformations and their circular specialization it is possible to derive all circular cubics of a plane and their properties.

UNIVERSITY OF ILLINOIS.

ON ANALYTIC FUNCTIONS OF CONSTANT MODULUS ON A GIVEN CONTOUR.*

By T. H. Gronwall.

§1. Introduction.

Let A be a simply-connected area in the plane of the complex variable x, and C the limiting contour of A. We assume C to be composed of a finite number of parts of analytic curves, or more generally, to be such as to allow the conformal representation of A on the interior of a circle. Let $f(x)$ be an analytic function uniform in A and on C, and having only a finite number of essential singularities $c_1, c_2, \cdots, c_n$ inside A and on C. Finally let P be an infinite set of points on C, having at least one limit point distinct from any of the points $c_1, \cdots, c_n$ which may lie on C, and suppose that, K being a given constant,

$$|f(x)| = K \tag{1}$$

in all points of P. It is required to find the general analytic expression for $f(x)$.

As stated above, A may be represented conformally on the interior of a circle, so that it is sufficient to consider the case of A being the interior of the circle

$$|x| = R. \tag{2}$$

In §2, it will first be shown that under the given assumptions equation (1) holds not only for the points of P, but for all points on the circumference (2), except for such of the essential singular points as may lie on it. Furthermore, the principle of analytic continuation due to H. A. Schwarz will be applied to the determination of all essential singularities, poles and zeros of the function. In §3, the general expression for $f(x)$ is given in the simplest case, where there are no essential singularities inside or on the circumference, and finally §4 gives the desired analytic expression in the general case.

§2. Preliminary investigation of the properties of $f(x)$.

According to our assumption, there exists at least one limit point a of the point-set P for which $|f(x)| = K$, such that $|a| = R$ and is not coincident with any of the essential singularities of $f(x)$.

* Presented to the American Mathematical Society Sept. 10, 1912.

Now the relation

(3) $$x = a\frac{1+iy}{1-iy}, \qquad y = i\frac{a-x}{a+x},$$

establishes a conformal representation of the interior of the circle (2) on the upper half plane, the circumference corresponding to the real axis, and $x = a$ to $y = 0$. In particular, the point-set P is transformed into a point-set Q on the real axis, one of the limit points of Q being $y = 0$.

Making

(4) $$g(y) = f\left(a\frac{1+iy}{1-iy}\right) = f(x),$$

it follows from the assumptions made regarding $f(x)$, that $g(y)$ is uniform in the upper half plane and on the real axis, having only a finite number of essential singularities $d_1, d_2, \cdots, d_n$ $\left(\text{where } d_\nu = i\dfrac{a-c_\nu}{a+c_\nu} \neq 0\right)$ in the area considered and on its boundary, and furthermore $|g(y)| = K$ on the point-set Q.

If $y = 0$ were a pole of $g(y)$, we would have $|g(y)| > 2K$ in a certain vicinity of $y = 0$, which is contrary to our assumption that $y = 0$ is a limit point of Q. We therefore have, for a certain radius r,

$$g(y) = \sum_{\nu=0}^{\infty}(\alpha_\nu + i\beta_\nu)y^\nu, \qquad |y| < r,$$

where α_ν and β_ν are real quantities. Writing

(5) $$\varphi(y) = \sum_{\nu=0}^{\infty}\alpha_\nu y^\nu, \qquad \psi(y) = \sum_{\nu=0}^{\infty}\beta_\nu y^\nu,$$

we have

(6) $$g(y) = \varphi(y) + i\psi(y), \qquad |y| < r,$$

and the condition $|g(y)| = K$ for the point-set Q gives

(7) $$\varphi(y)^2 + \psi(y)^2 = K^2$$

for the points of Q verifying the condition $|y| < r$. The left hand member of (7) being holomorphic for $|y| < r$, and $y = 0$ being a limit point of Q, it follows that (7) holds identically for $|y| < r$. From (6) and (7) we obtain

(8) $$\varphi(y) = \frac{1}{2}\left(g(y) + \frac{K^2}{g(y)}\right), \qquad \psi(y) = \frac{1}{2i}\left(g(y) - \frac{K^2}{g(y)}\right),$$

and from these equations, which define φ and ψ throughout the entire region of existence of $g(y)$, it is obvious that in the upper half plane and on the real axis, φ and ψ are uniform and have the essential singularities $d_1, \cdots, d_n$

only, and from (5) it follows that φ and ψ both assume real values on the part of the real axis given by $-r < y < r$.

Hence we may apply the principle of analytic continuation given by H. A. Schwarz,* and find that φ and ψ are uniform in the entire y-plane, having for essential singularities the points $d_1, \cdots, d_n$ and their conjugate points $d_1^0, \cdots, d_n^0$, and both assuming conjugate values for conjugate values of y. From the last property it follows that φ and ψ are both real in all points of the real axis (except for any of the essentially singular points which may lie on it), and consequently eq. (6) and (7), which now are proved valid in the entire y-plane, show *that* $|g(y)| = K$ *for all points of the real axis*, with the possible exception of some of the essential singularities. Hence $g(y)$ has neither zeros nor poles on the real axis. Equations (8) further show that a pole of either $\varphi(y)$ or $\psi(y)$ must be either a pole or a zero of $g(y)$ and conversely, that a pole or a zero of $g(y)$ must be a pole for both $\varphi(y)$ and $\psi(y)$. Thus the poles of $\varphi(y)$ are the same as those of $\psi(y)$ and coincide with the point-set formed by the zeros and poles of $g(y)$, and if $y = a$ belongs to this point-set, the same is the case with the conjugate point $y = a_0$. Let $y = a$ be a pole of $g(y)$ of the nth order; then a is a pole of the nth order of $\varphi(y)$ and $\psi(y)$, and these functions assuming conjugate values for conjugate values of y, $y = a_0$ must be a pole of the nth order of both $\varphi(y)$ and $\psi(y)$, that is, either a pole or a zero of the nth order of $g(y)$. Suppose a_0 to be a pole of $g(y)$; then we have the developments

$$g(y) = \frac{A}{(y-a)^n} + \cdots, \qquad g(y) = \frac{B}{(y-a_0)^n} + \cdots,$$

whence according to (8)

$$2\varphi(y) = \frac{A}{(y-a)^n} + \cdots, \qquad 2\varphi(y) = \frac{B}{(y-a_0)^n} + \cdots,$$

$$2\psi(y) = \frac{1}{i}\cdot\frac{A}{(y-a)^n} + \cdots, \qquad 2\psi(y) = \frac{1}{i}\cdot\frac{B}{(y-a_0)^n} + \cdots,$$

and as $\varphi(y)$ and $\varphi(y_0)$, $\psi(y)$ and $\psi(y_0)$ are conjugate quantities, we must have

$$B = A_0,$$

$$\frac{B}{i} = \left(\frac{A}{i}\right)_0 = -\frac{A_0}{i}, \qquad \text{or} \qquad B = -A_0,$$

whence $B = A_0 = A = 0$ contrary to our assumption. Therefore a_0 is a zero of the nth order of $g(y)$, and in the same way it is shown that if $y = a$ is a zero of the nth order of $g(y)$, $y = a_0$ is a pole of the nth order.

* H. A. Schwarz, "Über einige Abbildungsaufgaben," Journ. f. Math., *70* (1869), 105–120. See p. 107 of this paper or G. Darboux, Théorie des Surfaces, vol. I (1887), 174–176.

Returning to the variable x, it is known that to two conjugate values of y there correspond two points $x = a$ and $x = a'$, which are connected through the transformation by reciprocal radii

$$a' = \frac{R^2}{a_0}, \qquad a = \frac{R^2}{a_0'}, \tag{9}$$

and for $|x| = R$, we have

$$\left|\frac{x - a}{x - a'}\right| = \frac{|a|}{R}. \tag{10}$$

From what we have shown for the y-plane it then follows that $f(x)$ exists and is uniform in the entire x-plane; that two reciprocal points c_ν and c_ν' are both, or neither, essential singularities of $f(x)$, and that $x = a$ being a zero of the nth order (or $x = b$ a pole of the nth order) inside the circle $|x| = R$, $x = a'$ is a pole of the nth order (or $x = b'$ a zero of the nth order) outside the circle, and vice versa.

§3. The analytic expression for $f(x)$ when there are no essential singularities.

We now consider the particularly simple case where $f(x)$ has no essential singularities inside or on the circle $|x| = R$. According to the preceding paragraph, $f(x)$ then has no essential singularities outside of $|x| = R$, so that $f(x)$ is a rational function. Let

$$a_1, a_2, \cdots, a_m$$

be the zeros, and

$$b_1, b_2, \cdots, b_n$$

the poles of $f(x)$ inside $|x| = R$, each written as many times as indicated by its order; making

$$f_1(x) = \frac{\prod_{\nu=1}^{m} \frac{R(x - a_\nu)}{a_\nu(x - a_\nu')}}{\prod_{\nu=1}^{n} \frac{R(x - b_\nu)}{b_\nu(x - b_\nu')}} \tag{11}$$

(where, for $a_\nu = 0$, the factor $R(x - a_\nu)/a_\nu(x - a_\nu')$ has to be replaced by x/R, the same remark applying to any $b_\nu = 0$), and

$$f(x) = f_1(x)f_2(x), \tag{12}$$

it is seen that $f_2(x)$, being rational and having neither zeros nor poles, must be a constant, and as further, according to (10),

$$|f_1(x)| = 1 \qquad \text{for} \qquad |x| = R,$$

we have $|f_2(x)| = K$ for $|x| = R$ and consequently for every x, so that

$f_2(x) = Ke^{\gamma i}$, where γ is real. We thus finally obtain

$$f(x) = Ke^{\gamma i} \prod_{\nu=1}^{m} \frac{R(x - a_\nu)}{a_\nu(x - a_\nu')} \cdot \prod_{\nu=1}^{n} \frac{b_\nu(x - b_\nu')}{R(x - b_\nu)}. \tag{13}$$

This result is known* in the particular case where it is supposed from the beginning that $f(x)$ has a constant modulus on the entire circumference and not only in a point-set P. It may also be obtained in the following manner after showing that $|f(x)| = K$ on the entire circumference $|x| = R$, but without investigating the properties of $f(x)$ outside the circle. Forming (11) and (12) as previously, it is seen at once that $f_2(x)$ has neither zeros nor poles for $|x| \gtreqless R$, and that

$$|f_2(x)| = K \quad \text{for} \quad |x| = R,$$

whence

$$\left|\frac{1}{f_2(x)}\right| = \frac{1}{K} \quad \text{for} \quad |x| = R.$$

Both $f_2(x)$ and $1/f_2(x)$ being holomorphic for $|x| \gtreqless R$, it follows that

$$|f_2(x)| \gtreqless K \quad \text{for} \quad |x| \gtreqless R,$$

$$\left|\frac{1}{f_2(x)}\right| \gtreqless \frac{1}{K} \quad \text{for} \quad |x| \gtreqless R,$$

so that

$$|f_2(x)| = K \quad \text{for} \quad |x| \gtreqless R,$$

whence it is easily seen that $f_2(x) = \text{const.} = Ke^{\gamma i}$.

§4. The analytic expression for $f(x)$ in the general case.

We first remark that, for $|x| = R$, x and R^2/x are conjugate quantities, and consequently also

$$\frac{A}{(x - a)^n} \quad \text{and} \quad \frac{A_0}{\left(\dfrac{R^2}{x} - a_0\right)^n}.$$

By a simple transformation of the latter expression, we see that, for $|x| = R$,

$$\frac{A}{(x - a)^n} \quad \text{and} \quad \frac{(-1)^n A_0 a'^n x^n}{R^{2n}(x - a')^n} \tag{14}$$

are conjugate quantities.

Furthermore, let us distribute the zeros of $f(x)$ inside $|x| = R$ into n groups

* See, for instance, O. Blumenthal, "Sur le mode de croissance des fonctions entières," Bull. Soc. Math. de France, *35* (1907), 213–232, where it is obtained on pp. 214–215 by the aid of harmonic functions.

$$\begin{array}{ccccc} a_{11}, & a_{12}, & \cdots, & a_{1\lambda}, & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1}, & a_{n2}, & \cdots, & a_{n\lambda}, & \cdots \end{array}$$

such that

$$\lim_{\lambda=\infty} a_{\nu\lambda} = c_\nu \qquad (\nu = 1, 2, \cdots, n),$$

and similarly the poles inside $|x| = R$:

$$\begin{array}{ccccc} b_{11}, & b_{12}, & \cdots, & b_{1\lambda}, & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1}, & b_{n2}, & \cdots, & b_{n\lambda}, & \cdots \end{array}$$

so that

$$\lim_{\lambda=\infty} b_{\nu\lambda} = c_\nu \qquad (\nu = 1,\ 2,\ \cdots\ n)\ *$$

We now form the primary factor

$$E(x; a_{\nu\lambda}, a_{\nu\lambda}'; c_\nu, c_\nu') = \frac{1 - \dfrac{a_{\nu\lambda} - c_\nu}{a_{\nu\lambda}} \cdot \dfrac{x}{x - c_\nu}}{1 - \dfrac{a_{\nu\lambda}' - c_\nu'}{x - c_\nu'}} \cdot e^{\sum_{\mu=1}^{n_\lambda - 1} \frac{1}{\mu}\left[\left(\frac{a_{\nu\lambda} - c_\nu}{a_{\nu\lambda}} \frac{x}{x - c_\nu}\right)^\mu - \left(\frac{a_{\nu\lambda}' - c_\nu'}{x - c_\nu'}\right)^\mu\right]},$$

where n_λ is an integer to be determined later.

For $|x| = R$, we have by (10)

$$\left| \frac{1 - \dfrac{a_{\nu\lambda} - c_\nu}{a_{\nu\lambda}} \dfrac{x}{x - c_\nu}}{1 - \dfrac{a_{\nu\lambda}' - c_\nu'}{x - c_\nu'}} \right| = \left| \frac{c_\nu}{a_{\nu\lambda}} \frac{x - c_\nu'}{x - c_\nu} \frac{x - a_{\nu\lambda}}{x - a_{\nu\lambda}'} \right| = \left| \frac{c_\nu}{a_{\nu\lambda}} \right| \cdot \frac{R}{|c_\nu|} \cdot \frac{|a_{\nu\lambda}|}{R} = 1,$$

and according to (14), the two terms inside the bracket in the exponential are conjugate quantities for $|x| = R$. Therefore we have

(16) $$|E(x; a_{\nu\lambda}, a_{\nu\lambda}'; c_\nu, c_\nu')| = 1 \qquad \text{for} \qquad |x| = R.$$

In the case $c_\nu = 0$, $c_\nu' = \infty$, the expression (15) should be replaced by

(15a) $$E(x; a_{\nu\lambda}, a_{\nu\lambda}'; 0, \infty) = \frac{1 - \dfrac{a_{\nu\lambda}}{x}}{1 - \dfrac{x}{a_{\nu\lambda}'}} e^{\sum_{\mu=1}^{n_\lambda - 1} \frac{1}{\mu}\left[\left(\frac{a_{\nu\lambda}}{x}\right)^\mu - \left(\frac{x}{a_{\nu\lambda}'}\right)^\mu\right]},$$

where we obviously have

(16a) $$|E(x; a_{\nu\lambda}, a_{\nu\lambda}'; 0, \infty)| = 1 \qquad \text{for} \qquad |x| = R,$$

and in the case of $a_{\nu\lambda} = 0$, $a_{\nu\lambda}' = \infty$ we write

(15b) $$E(x; 0, \infty; c_\nu, c_\nu') = \frac{x}{R},$$

* In the case where c_ν is not a limit point of poles or zeros, the corresponding primary factors should be suppressed in equations (15) to (17).

so that

$$(16b) \qquad |E(x;\, 0,\, \infty;\, c_\nu,\, c_\nu')| = 1 \qquad \text{for} \qquad |x| = R.$$

It is shown in the classical way,* that by conveniently determining n_λ (for instance, $n_\lambda = \lambda$), the infinite product

$$\prod_{\lambda=1}^{\infty} E(x;\, a_{\nu\lambda},\, a_{\nu\lambda}';\, c_\nu,\, c_\nu')$$

is uniformly convergent in any part of the plane for which c_ν, c_ν' and $a_{\nu 1}'$, $a_{\nu 2}'$, $\cdots$ are exterior points, and that the product represents a uniform analytic function of x, having c_ν and c_ν' for its essential singularities, $a_{\nu 1}$, $a_{\nu 2}$, $\cdots$ for its zeros, $a_{\nu 1}'$, $a_{\nu 2}'$, $\cdots$ for its poles, and finally of modulus $= 1$ for $|x| = R$.
Now make

$$(17) \qquad f(x) = \prod_{\nu=1}^{n} \prod_{\lambda=1}^{\infty} E(x;\, a_{\nu\lambda},\, a'_{\nu\lambda};\, c_\nu,\, c_\nu') \cdot f_1(x);$$

then $f_1(x)$ has neither zeros, nor poles, nor essential singularities outside of the points c_ν, c_ν' ($\nu = 1, 2, \cdots, n$), and furthermore

$$|f_1(x)| = K \quad \text{for} \quad |x| = R.$$

Let $c_1, c_2, \cdots, c_m$ be those of the points $c_1, \cdots, c_n$ that lie inside, and $c_{m+1}, \cdots, c_n$ those that lie on the circumference $|x| = R$; as obviously $c_\nu' = c_\nu$ for $\nu = m + 1, \cdots, n$, the only possible zeros or singularities of $f_1(x)$ are $c_1, \cdots, c_m;\, c_1', \cdots, c_m';\, c_{m+1}, \cdots, c_n$.

In the vicinity of $x = c_\nu$ ($\nu = 1, 2, \cdots, n$) we have the following development (Weierstrass, l. c.):

$$(19) \qquad f_1(x) = (x - c_\nu)^{l_\nu} e^{G_\nu\left(\frac{1}{x - c_\nu}\right) + P_\nu(x - c_\nu)},$$

where l_ν is an integer and

$$(20) \qquad G_\nu\left(\frac{1}{x - c_\nu}\right) = \sum_{\lambda=1}^{\infty} A_{\nu\lambda}\left(\frac{1}{x - c_\nu}\right)^{\lambda}$$

an integral function of its argument. Writing

$$(21) \qquad \varphi_\nu(x) = \left(\frac{R}{c_\nu}\,\frac{x - c_\nu}{x - c_\nu'}\right)^{l_\nu} e^{\sum_{\lambda=1}^{\infty}\left[\frac{A_{\nu\lambda}}{(x - c_\nu)^\lambda} + \frac{(-1)^{\lambda+1}(A_{\nu\lambda})_0 c_\nu'^{\lambda} x^\lambda}{R^{2\lambda}(x - c_\nu')^\lambda}\right]},$$

* See, for instance, K. Weierstrass, "Zur Theorie der eindeutigen analytischen Funktionen," Abhandl. d. Ak. d. Wiss. Berlin, 1876, 11–60, or Abhandlungen aus der Funktionenlehre, 1–52, or Math. Werke, II, 77–124; G. Mittag-Leffler, "Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante," Acta Math., *4* (1884), 1–79.

or, respectively, for $c_\nu = 0$,

$$\varphi_\nu(x) = \left(\frac{x}{R}\right)^{l_\nu} e^{\sum\limits_{\lambda=1}^{\infty}\left[\frac{A_{\nu\lambda}}{x^\lambda} - \frac{(A_{\nu\lambda})_0 x^\lambda}{R^{2\lambda}}\right]}, \tag{21a}$$

$\varphi_\nu(x)$ has no zeros or singularities except c_ν, c_ν', and by (10) and (14)

$$|\varphi_\nu(x)| = 1 \qquad \text{for} \qquad |x| = R. \tag{22}$$

Now let us make

$$f_1(x) = \varphi_1(x)\varphi_2(x) \cdots \varphi_m(x) \cdot f_2(x); \tag{23}$$

then, by (19) and (21), $f_2(x)$ is holomorphic and different from zero for $x = c_\nu$ $(\nu = 1, 2, \cdots, m)$, and by (18) and (22),

$$|f_2(x)| = K \quad \text{for} \quad |x| = R. \tag{24}$$

Accordingly, $f_2(x)$ is also holomorphic and different from zero for $x = c_\nu'$ $(\nu = 1, 2, \cdots, m)$, so that the only possible singularities or zeros of $f_2(x)$ are $x = c_{m+1}, \cdots, c_n$, and in the vicinity of each of these we have, by (19) and (23),

$$f_2(x) = (x - c_\nu)^{l_\nu} e^{G_\nu\left(\frac{1}{x-c_\nu}\right) + P_{\nu 1}(x-c_\nu)} \qquad (\nu = m+1, \cdots, n). \tag{25}$$

Now make the conformal representation

$$y = i\frac{c_\nu - x}{c_\nu + x};$$

then in the vicinity of $y = 0$ (corresponding to $x = c$) we have

$$\log f_2(x) = l_\nu \log\left(\frac{-2c_\nu y}{y+i}\right) + \sum_{\lambda=1}^{\infty} \frac{B_{\nu\lambda} + iC_{\nu\lambda}}{y^\lambda} + \sum_{\lambda=0}^{\infty} (\bar{B}_{\nu\lambda} + i\bar{C}_{\nu\lambda})y^\lambda,$$

where all B and C are real quantities. For y real (corresponding to $|x| = R$) the real part of this expression is

$$\tfrac{1}{2} l_\nu \log\frac{y^2}{y^2+1} + l_\nu \log(2R) + \sum_{\lambda=1}^{\infty} \frac{B_{\nu\lambda}}{y^\lambda} + \sum_{\lambda=0}^{\infty} \bar{B}_{\nu\lambda} y^\lambda,$$

and as, according to (24), this is equal to $\log K$, we must have

$$l_\nu = 0, \qquad B_{\nu\lambda} = \bar{B}_{\nu\lambda} = 0 \quad \begin{pmatrix} \lambda = 1, 2, 3, \cdots \\ \nu = m+1, \cdots, n \end{pmatrix}.$$

Consequently, making

$$\varphi_\nu(x) = e^{\sum\limits_{\lambda=1}^{\infty} iC_{\nu\lambda}\left(\frac{c_\nu + x}{i(c_\nu - x)}\right)^\lambda} = e^{\sum\limits_{\lambda=1}^{\infty} i^{\lambda+1} C_{\nu\lambda}\left(\frac{x + c_\nu}{x - c_\nu}\right)^\lambda} \qquad (\nu = m+1, \cdots, n), \tag{26}$$

φ_ν has neither zeros nor poles, its only possible essential singularity is c_ν, and

$$|\varphi_\nu(x)| = 1 \quad \text{for} \quad |x| = R. \tag{27}$$

Now writing

$$f_2(x) = \varphi_{m+1}(x) \cdots \varphi_n(x) \cdot f_3(x), \tag{28}$$

$f_3(x)$ has no singularities and therefore is a constant; furthermore $|f_3(x)| = K$ for $|x| = R$, so that finally

$$f_3(x) = Ke^{\gamma i}. \tag{29}$$

We have thus solved the proposed problem, and it is evident that the method used may be extended to the case where the essential singularities of $f(x)$ inside or on $|x| = R$, instead of being finite in number, form a point-set belonging to the general class considered by Mittag-Leffler in the paper previously quoted.

NECESSARY AND SUFFICIENT CONDITIONS FOR THE INTERCHANGE OF LIMIT AND SUMMATION IN THE CASE OF SEQUENCES OF INFINITE SERIES OF A CERTAIN TYPE.

By T. H. Hildebrandt.

Double sequences and series have been discussed by Pringsheim,* London,† and others. They have treated incidentally the question of interchange of iterated limits. An interchange of limit and infinite summation, although to some extent a question of interchange of limits, is a distinct problem in that summation and limit are different operations. The theorems of this paper derive necessary and sufficient conditions for the interchange of limit in the case of a special type of series, practically that of series of positive terms.

Theorem I. *Suppose a double sequence of numbers* x_{np} $(n = 1, 2, \cdots;$ $p = 1, 2, \cdots)$ *such that* $\Sigma_p|x_{np}|$ ‡ *is convergent for every* n. *Suppose also* $L_n x_{np} = x_p$ *for every* p. *Then a necessary and sufficient condition that* $\Sigma_p|x_p|$ *converge and* $L_n\Sigma_p|x_{np}| = \Sigma_p|x_p|$ *is that the series* $\Sigma_p|x_{np}|$ *be uniformly convergent.*

That the condition is *sufficient* even in case the absolute value signs be dropped throughout is well known, a consequence of the theorem on the interchange of double limits.§

On the other hand the condition is *necessary*. Since $\Sigma_p|x_p|$ converges we have for every $e > 0$ a p_e such that if $P \geqq p_e$ then:

$$\sum_{p=P}^{\infty} |x_p| \leqq e/2.$$

Take a particular value of P, say $p_1 = p_e$. Then from

$$\underset{n}{L} \sum_{p} |x_{np}| = \sum_{p} |x_p| \qquad \text{follows} \qquad \underset{n}{L} \sum_{p=p_1}^{\infty} |x_{np}| = \sum_{p=p_1}^{\infty} |x_p|,$$

i. e., for every e there exists an n_e such that if $n \geqq n_e$ we have:

$$\left| \sum_{p=p_1}^{\infty} |x_{np}| - \sum_{p=p_1}^{\infty} |x_p| \right| \leqq e/2.$$

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* Pringsheim, Muench. Ber. (1897), pp. 101–152; Math. Ann., 53 (1900), pp. 289–321.

† London, Math. Ann., 53 (1900), pp. 322–370.

‡ Throughout this paper we shall designate $\sum_{p=1}^{\infty}$ by $\sum_{p}$ and $\underset{n=\infty}{L}$ by $\underset{n}{L}$.

§ Cf., for instance, Hobson, Theory of Functions, p. 466.

Then if $P \gtreqless p_1 = p_e$ and $n \gtreqless n_e$, we have:

$$\sum_{p=P}^{\infty} |x_{np}| \leqq \sum_{p=p_1}^{\infty} |x_{np}| \lesseqgtr \sum_{p=p_1}^{\infty} |x_p| + e/2 \lesseqgtr e/2 + e/2 = e.$$

Since there will be a finite number of values of n less than n_e we have: for every $e > 0$ there exists a p_e' such that for $P \gtreqless p_e'$ we have:

$$\sum_{p=P}^{\infty} |x_{np}| \lesseqgtr e, \qquad n < n_e.$$

Hence if p_e'' is the greater of p_e and p_e' we have for every n and for every $e > 0$ there exists a p_e'' such that for $P \gtreqless p_e''$ we have:

$$\sum_{p=P}^{\infty} |x_{np}| \lesseqgtr e;$$

which is the uniformity of convergence desired.

COROLLARY. *If $m > 0$, $\Sigma_p |x_{np}|^m$ is convergent for every n and $L_n x_{np} = x_p$ for every p, then a necessary and sufficient condition that $\Sigma_p |x_p|^m$ be convergent and $L_n \Sigma_p |x_{np}|^m = \Sigma_p |x_p|^m$ is that $\Sigma_p |x_{np}|^m$ be uniformly convergent.*

THEOREM II. *Suppose $m > 0$, $\Sigma_p |x_{np}|^m$ convergent for every n, and $L_n x_{np} = x_p$ for every p. Then a necessary and sufficient condition that $\Sigma_p |x_p|^m$ be convergent and $L_n \Sigma_p |x_{np}|^m = \Sigma_p |x_p|^m$, is that $L_n \Sigma_p |x_{np} - x_p|^m = 0$.*

To prove this theorem we make use of the following inequality *

$$\left[\sum_{p=1}^{n} |a_p + b_p|^m\right]^k \lesseqgtr \left[\sum_{p=1}^{n} |a_p|^m\right]^k + \left[\sum_{p=1}^{n} |b_p|^m\right]^k, \tag{1}$$

where $m > 0$, and $k = 1/m$ if $m > 1$ and $k = 1$ if $m < 1$. This inequality may be extended to n infinite if the series on the right hand side are convergent. We shall refer to this extension as inequality (1′).

The condition of the theorem is *necessary*. Since $\Sigma_p |x_p|^m$ is convergent and $L_n \Sigma_p |x_{np}|^m = \Sigma_p |x_p|^m$, we have by the corollary above that $\Sigma_p |x_{np}|^m$ are uniformly convergent, i. e., for every $e > 0$ there exists a p_e such that if $P \gtreqless p_e$, we have:

$$\sum_{p=P}^{\infty} |x_{np}|^m \lesseqgtr e^{1/k} \qquad \text{and} \qquad \sum_{p=P}^{\infty} |x_p|^m \lesseqgtr e^{1/k}.$$

Take $P = p_1$ fixed. For $p = 1, 2, \cdots, p_1$, we have, since $L_n x_{np} = x_p$ for every p: for every $e > 0$ there exists an n_e such that if $n \gtreqless n_e$ we have:

$$|x_{np} - x_p|^m \lesseqgtr (e/p_1)^{1/k}. \tag{2}$$

* Cf. Riess, Math. Ann., vol. 69 (1910), p. 455.

Using the inequality (1′) and the fact that $k \overline{\overline{<}} 1$ for every m, we have:

$$\left[\sum_p |x_{np} - x_p|^m\right]^k \overline{\overline{<}} \sum_{p=1}^{p_1}\left[|x_{np} - x_p|^m\right]^k + \left[\sum_{p=p_1}^{\infty}|x_{np} - x_p|^m\right]^k$$

$$\overline{\overline{<}}\, e + \left[\sum_{p=p_1}^{\infty}|x_{np}|^m\right]^k + \left[\sum_{p=p_1}^{\infty}|x|^m\right]^k \overline{\overline{<}}\, 3e,$$

i. e., for every $e > 0$ there exists an n_e, viz., the n_e needed for inequality (2), such that if $n \overline{\overline{>}} n_e$, we have:

$$\left[\sum_{p=1}^{\infty}|x_{np} - x_p|^m\right]^k \overline{\overline{<}}\, e,$$

in other words:

$$\underset{n}{L}\sum_p |x_{np} - x_p|^m = 0.$$

The condition is *sufficient*.* Since $L_n\Sigma_p|x_{np} - x_p|^m = 0$, for every $e > 0$ there will exist an n_e such that if $n \overline{\overline{>}} n_e$ we have:

$$\sum_p |x_{np} - x_p|^m \overline{\overline{<}}\, e. \tag{3}$$

Let $x_{np} - x_p = e_{np}$. Then $x_p = x_{np} + e_{np}$ and $x_{np} = x_p - e_{np}$. Applying inequality (1′), we obtain:

$$[\sum_p |x_p|^m]^k = [\sum_p |x_{np} + e_{np}|^m]^k \overline{\overline{<}} [\sum_p |x_{np}|^m]^k + [\sum_p |e_{np}|^m]^k$$

and

$$[\sum_p |x_{np}|^m]^k = [\sum_p |x_p - e_{np}|^m]^k \overline{\overline{<}} [\sum_p |x_p|^m]^k + [\sum_p |e_{np}|^m]^k.$$

From the first of these inequalities we conclude that $\sum_p |x_p|^m$ is convergent.† From the two inequalities taken together and the condition $n \overline{\overline{>}} n_e$, which gives us inequality (3), we have:

$$[\sum_p |x_{np}|^m]^k - e^k \overline{\overline{<}} [\sum_p |x_p|^m]^k \overline{\overline{<}} [\sum_p |x_{np}|^m]^k + e^k$$

and so

$$\underset{n}{L}\sum_p |x_{np}|^m = \sum_p |x_p|^m.$$

ANN ARBOR, MICH.,
March, 1912.

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* For $m = 2$ this is a consequence of a theorem due to Hilbert, Goett. Nach. Math. Phys. Klasse (1906), p. 177.

† Also a consequence of Moore, General Analysis (Yale Coll. Lect.) §16, p. 38.

A SIMPLE PROOF OF A FUNDAMENTAL THEOREM IN THE THEORY OF INTEGRAL EQUATIONS.*

By Maxime Bôcher.

Let $\varphi_1(x), \cdots, \varphi_n(x)$ be functions of the real variable x which throughout the interval

$$I \qquad a \overline{\overline{<}} x \overline{\overline{<}} b$$

are continuous, or, more generally, whose absolute values together with their squares are integrable. These functions, and the others with which we shall deal, need not be real, and we will indicate the conjugate imaginary function by a dash placed above the symbol for the function.

The n-rowed determinant in which the element in the ith row and jth column is

$$\int_a^b \varphi_i(x)\bar{\varphi}_j(x)dx$$

we call the Gramian of $\varphi_1, \cdots, \varphi_n$ and denote it by $G(\varphi_1, \cdots, \varphi_n)$. It has the property† of being always real and positive except when $\varphi_1, \cdots, \varphi_n$ are linearly dependent,‡ in which case it is equal to zero.

Let us denote by G_ϕ the $(n+1)$-rowed determinant obtained by bordering G on the right by the column $\varphi_1, \cdots, \varphi_n$, 0 and at the bottom by the row $\bar{\varphi}_1, \cdots, \varphi_n$, 0. If we expand this determinant according to the elements of the last column, it takes the form of the sum of n determinants of the nth order each of which differs from $G(\varphi_1, \cdots, \varphi_n)$ only in having the integral signs omitted from the elements of one row and in having its sign reversed. Consequently the integral of each of these determinants is $-G(\varphi_1, \cdots, \varphi_n)$, and we have the formula

$$(1) \qquad \int_a^b G_\phi dx = -nG(\varphi_1, \cdots, \varphi_n).$$

Suppose now that $\varphi_1, \cdots, \varphi_n$ form a set of linearly independent continuous solutions of the integral equation

$$(2) \qquad u(x) = \int_a^b K(x, \xi)u(\xi)d\xi,$$

* Presented to the American Mathematical Society December 28, 1910.

† Cf. E. Fischer, Archiv der Mathematik und Physik, third series, vol. 13 (1908), p. 32, or Kowalewski, Einführung in die Determinantentheorie, p. 333.

‡ We say that $\varphi_1, \cdots, \varphi_n$ are linearly dependent when and only when there exist n constants $c_1, \cdots, c_n$ not all zero such that $c_1\varphi_1 + \cdots + c_n\varphi_n$ vanishes at all points of I except perhaps at the points of a set of content zero.

where $|K|$ and $|K^2|$ are integrable functions of the two independent variables (x, ξ) so long as x and ξ both lie in I. The inequality

$$G[\varphi_1(\xi), \cdots, \varphi_n(\xi), \overline{K}(x, \xi)] \geqq 0$$

may be written as follows when we remember that the φ's are solutions of (2):

$$G(\varphi_1, \cdots, \varphi_n) \int_a^b K(x, \xi)\overline{K}(x, \xi)d\xi + G_\phi \geqq 0.$$

Consequently, if we integrate with regard to x from a to b, make use of (1), and throw out the positive factor $G(\varphi_1, \cdots, \varphi_n)$, we get:

$$n \overline{\overline{<}} \int_a^b \int_a^b |K(x, \xi)|^2 d\xi\, dx. \tag{3}$$

Thus we have proved the theorem*

If $|K|$ *and its square are integrable functions of* (x, ξ), *the number* n *of linearly independent continuous solutions of* (2) *is limited by* (3).

The possibility of immediate extension to any number of variables is obvious.

BUNTENBOCK IM HARZ,
July, 1910.

* For another proof of this theorem see I. Schur, Mathematische Annalen, vol. 66 (1909), p. 508.

AN APPLICATION OF MODULAR EQUATIONS IN ANALYSIS SITUS.*

By Oswald Veblen.

1. By a map we mean a set of α_2 simply connected regions (countries) covering the surface of a sphere and bounded by α_1 simple arcs (edges) joining α_0 distinct points (vertices). No two regions have a point in common, no two arcs intersect.

For all known maps it is possible to assign to each region one of four colors in such a way that any two regions having an edge in common are differently colored. Whether or not this is true for all maps is still unknown in spite of the investigations of a considerable number of mathematicians.† It is therefore perhaps not without interest to show how the problem can be stated in terms of linear equations in a finite field.

These equations turn out to be of service in describing the elementary properties of the map. In particular they supply us with an easy proof of Euler's formula. We shall first outline the discussion of these equations as they arise in the field of integers reduced modulo two and then show how they connect with the four-color problem when the field is extended to include certain Galois imaginaries.

2. A map can be fully described by means of two matrices.‡ To do this, the vertices are numbered in an arbitrary way from 1 to α_0, the edges from 1 to α_1, and the countries from 1 to α_2. In the first matrix the rows correspond to the vertices and the columns to the edges. A " 1 " appears as the element of the ith row and jth column if the ith vertex is on the jth edge; and a " 0 " appears as the element of the ith row and jth column if the ith vertex is not on the jth edge. We shall denote this matrix by A; it has α_0 rows and α_1 columns. In the second matrix the rows correspond to the edges and the columns to the countries. The element of the ith

* Read before the American Mathematical Society, April 27, 1912.

† A fairly complete set of references is to be found in the article on "Analysis Situs" in the Encyklopädie der mathematischen Wissenschaften, III, AB 3, p. 177. The most recent paper on the subject, "On the Reducibility of Maps" by G. D. Birkhoff will appear in the American Journal of Mathematics, Vol. 35 (1913).

‡ These matrices are identical on interchanging rows and columns with those employed by Poincaré, Proceedings of the London Mathematical Society, vol. 32, 1900, p. 277, if the + and − signs used by the latter are omitted. The + and − signs are clearly not essential to the determination of a manifold.

row and jth column is 1 if the ith edge is on the jth region and "0" if not. We shall call this matrix B; it has α_1 rows and α_2 columns.

For the map obtained by projecting an inscribed tetrahedron from one of its interior points to the surface of a sphere the matrixes A and B are respectively (cf. Fig. 1):

$$A = \begin{Vmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{Vmatrix}, \qquad B = \begin{Vmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{Vmatrix}.$$

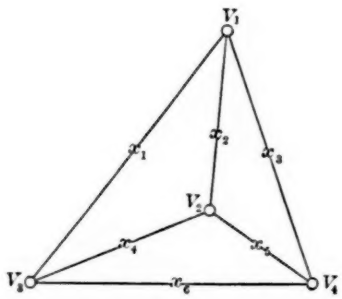

FIG. 1.

3. With these matrices may be associated four sets of linear homogeneous equations. In each case the variables and coefficients are regarded as integers reduced modulo two. In other words, let us add according to the rules $1 + 1 = 0$, $1 + 0 = 1$, $0 + 1 = 1$, $0 + 0 = 0$; and multiply according to the rules $1 \times 1 = 1$, $1 \times 0 = 0$, $0 \times 1 = 1$, $0 \times 0 = 0$. All the formal laws of elementary algebra are satisfied by this field.

In the first set the equations correspond to the rows of the matrix A. There is one variable for each edge of the map and one equation,

$$(1) \qquad x_a + x_b + x_c + \cdots = 0$$

for each vertex, the variables in the equation representing the edges which meet at the corresponding vertex. A solution of this system of equations represents a way of labelling the edges of the map with 0's and 1's so that there shall be an even number of 1's on the edges at each vertex. *The edges labelled with 1's in this manner form a number of closed circuits no two of which have an edge in common.* For let us start with an arbitrary edge labelled 1 and describe a path among the edges labelled 1. Whenever

there is an edge by which this path approaches a vertex, since the number of 1-edges at this vertex is even, there is a 1-edge by which the path can go away. Hence the path may be continued till it intersects itself. A portion of the path then forms a closed circuit. If this be removed there are still an even number of 1-edges at each vertex. Another circuit may be removed and so on till all the 1-edges are accounted for.

If $(x_1, x_2, \cdots, x_a)$ and $(x_1', x_2', \cdots, x_a')$ are solutions of the equations (1) it is clear that $(x_1 + x_1', x_2 + x_2', \cdots, x_{a_1} + x_{a_1}')$ is also a solution. The boundary of each of the α_2 countries of the map is represented by a solution in which each edge of the boundary is marked with a 1 and each other edge with a 0. One such solution is supplied by each column of the matrix B. We shall call these the *fundamental solutions*. The solution representing any circuit whatever may be expressed linearly in terms of these α_2 fundamental solutions. In fact, the circuit divides the surface of the sphere into two parts, and the solution representing the circuit is expressible as the sum of the solutions corresponding to the countries in one of these parts.

The sum of the α_2 fundamental solutions is $(0, 0, \cdots, 0)$ because each edge appears on the boundary of two and only two countries. These solutions cannot be subject to any other linear homogeneous relation because the coefficients of such a relation could be only 0 or 1, and hence the relation would merely state that the sum of a certain subset of solutions would be $(0, 0, \cdots, 0)$. This is impossible because any subset of the α_2 countries has at least one country with an edge not on any other country of the subset. Hence *the number of linearly independent solutions of the equations* (1) *is* $\alpha_2 - 1$, *and the total number of solutions is* 2^{α_2-1}.

4. A second set of equations is determined by the columns of the first matrix. In these equations the variables correspond to the vertices of the map and there is one equation

$$v_a + v_b = 0 \tag{2}$$

for each edge, v_a and v_b being the vertices at the ends of the edge. The only possible solutions are such that all the variables are equal. For if v_a is given, v_b must be equal to v_a; if v_c is connected with v_b by an edge, v_c is also equal to v_a, and so on. Since there is a path along the edges joining any vertex to any other it follows by this argument that all the variables are equal to v_a. Hence the only solutions of the equations are $(0, 0, \cdots, 0)$ and $(1, 1, \cdots, 1)$. There are α_0 variables. Hence the number of the α_1 equations which are linearly independent must be $\alpha_0 - 1$. *Hence the rank of the matrix A is $\alpha_0 - 1$.*

5. Let us return a moment to the first set of equations. Here there

were α_0 equations among α_1 variables and there were $\alpha_2 - 1$ solutions. The rank of the matrix A has just been seen to be $\alpha_0 - 1$ so that the number of linearly independent equations is $\alpha_0 - 1$. Hence

$$\alpha_1 - (\alpha_0 - 1) = \alpha_2 - 1,$$

or

$$\alpha_0 - \alpha_1 + \alpha_2 = 2,$$

which is the well-known Euler's formula.

6. The third set of equations may be read from the rows of the matrix B. The variables correspond to the countries and for each edge there is an equation of the form

$$y_a + y_b = 0, \tag{3}$$

where y_a and y_b correspond to the countries meeting in the edge in question. These equations are entirely analogous to the equations (2). In fact if a point (the capital of the country, as Mr. Bennett suggests calling it) be introduced in each region and the points in abutting regions be joined by non intersecting arcs, there is obtained a map (of α_2 points, α_1 edges and α_0 regions) dual to the first map and interchanging the rôles of the matrices A and B. The only solutions of the equations (3) are, by precisely the argument used for the equations (2), $(0, 0, \cdots, 0)$ and $(1, 1, \cdots, 1)$. Hence the rank of B is $\alpha_2 - 1$.

7. The fourth set of equations correspond to the columns of the matrix B. The variables correspond to the edges of the map and for each country there is an equation of the form

$$e_a + e_b + \cdots = 0, \tag{4}$$

where e_a, e_b, $\cdots$ are the edges of the region. Just as in the case of the equations (1), a solution represents a system of circuits among the regions of the map. A circuit is a set of distinct countries $r_1, r_2, \cdots, r_n$ and distinct edges $e_1, e_2, \cdots, e_n$ such that r_1 and r_2 meet along e_1, r_2 and r_3 meet along e_2, $\cdots$, r_n and r_1 meet along e_n. A circuit is *simple* if no subset of its edges and regions form a circuit. To distinguish a simple circuit composed of edges and regions from an ordinary circuit which is composed of edges and vertices we may call the former a *cycle*. The set of all edges and regions meeting at a vertex form a cycle which we shall call *fundamental*. The solutions which correspond to fundamental cycles are given by the rows of Matrix A of which $\alpha_0 - 1$ are linearly independent. Since the rank of B is $\alpha_2 - 1$ and the number of variables in the equations (4) is α_1, the number of linearly independent solutions is

$\alpha_1 - \alpha_2 + 1$, which by Euler's formula is $\alpha_0 - 1$. Hence the α_0 fundamental cycles furnish a set of solutions of (4) in terms of which all the solutions are expressible; and these fundamental solutions satisfy one linear homogeneous relation.

8. Turning to the four color problem, let us suppose the field $GF(2)$, consisting of 0 and 1 combined modulo two, to be extended by the Galois imaginaries satisfying the relations $i^2 + i + 1 = 0$. The extended field $GF(2^2)$ has four elements, $0, 1, i, i+1$ which we may use to denote the four colors. Two elements α, β of this field are equal if and only if $\alpha+\beta=0$. *Hence a solution of the four color problem consists in finding a set of values* $(y_1, y_2, \cdots, y_{\alpha_2})$ *which satisfies none of the equations* (3) *corresponding to the rows of the matrix B.*

9. The set of values $(y_1, y_2, \cdots, y_{\alpha_2})$ may be regarded as a point in a finite projective space of $\alpha_2 - 1$ dimensions* provided we exclude the set $(0, 0, 0, \cdots, 0)$ and regard $(ky_1, ky_2, \cdots ky_{\alpha_2})$ as the same as $(y_1, y_2, \cdots, y_{\alpha_2})$. Each of the equations (3) then represents an $(\alpha_2 - 2)$-space. If the variables y_1 range only over the $GF(2)$ there will be $2^{\alpha_2}-1$ points in the (α_2-1) space. If the variables range over the $GF(2^2)$ there will be $(4^{\alpha_2} - 1)/3$ such points. The first space is included in the second and the points of the second space not included in the first may be regarded as *imaginary* with respect to the first space.

In general, there can be no real point which satisfies none of the equations (3) for such a point would represent a coloring of the map by two colors, which is impossible whenever an odd number of regions meet at any vertex. Hence, in general, every real point lies on at least one of the $(\alpha_2 - 2)$-spaces.

An imaginary point can be written in the form $(y_1 + iy_1', y_2 + iy_2', \cdots, y_{\alpha_2}+iy_{\alpha_2}')$ where $(y_1, y_2, \cdots, y_{\alpha_2})$ and $(y_1', y_2', \cdots, y_{\alpha_2}')$ are real. Hence every imaginary point is on a real line. No imaginary point can be on two real lines because two such lines if they intersect at all have a real point in common. But if $(y_1 + iy_1', y_2 + iy_2', \cdots, y_{\alpha_2} + iy_{\alpha_2}')$ satisfies one of the equations (3), so must $(y_1, y_2, \cdots, y_{\alpha_2})$ and $(y_1', y_2', \cdots, y_{\alpha_2}')$, and conversely. Hence a solution of the four color problem is given *by each real line which does not lie on any of the* $(\alpha_2 - 2)$ *spaces which are represented by equations* (3).

10. If a point of the real $(\alpha_2 - 1)$-space does not satisfy any of the equations (3) corresponding to the edges in a cycle (cf. §7) the countries in the cycle must be assigned alternately the values 0 and 1. This is impossible in a cycle containing an odd number of regions. Hence *every*

* Cf. Veblen and Bussey, Transactions of the American Mathematical Society, vol. 7 (1906), p. 240.

real point $(y_1, y_2, \cdots, y_{\alpha_2})$ *lies in at least one* $(\alpha_2 - 2)$*-space corresponding to an edge of each odd cycle in the map.*

If the equations corresponding to the edges in a cycle be added, each variable enters twice in the sum. Hence the sum vanishes. In other words, the equations corresponding to the edges in the cycle satisfy a linear relation. If any subset of these equations be added it is clear from the definition of a cycle that there is at least one variable which enters only once in the sum, and hence the sum does not vanish. Hence the equations corresponding to the edges in a cycle satisfy only one linear relation. A set of n $(\alpha_2 - 2)$-spaces in an $(\alpha_2 - 1)$-space would in general meet in an $(\alpha_2 - n - 1)$-space. But the $(\alpha_2 - 2)$ spaces corresponding to the edges in a cycle satisfy one linear relation and therefore meet in an $(\alpha_2 - n)$-space. Let us denote by $S^c_{\alpha_2 - n}$ the $(\alpha_2 - n)$-space thus determined by a cycle C_n of n regions.

Any point of $S^c_{\alpha_2 - n}$ must satisfy all the equations corresponding to edges of C_n. Hence, if n is odd, a line joining any point of $S^c_{\alpha_2 - n}$ to any point whatever must lie in at least one of the $(\alpha_2 - 2)$-spaces corresponding to the edges in C_n. Therefore a line which furnishes a solution of the four color problem cannot pass through any point of the $S^c_{\alpha_2 - n}$ corresponding to any odd chain C_n. In order that it be possible to color the map it is necessary that there be at least one point $(y_1, y_2, \cdots, y_{\alpha_2})$ not on any $S^c_{\alpha_2 - n}$ for which n is odd.

This condition is also sufficient. For suppose we have a point $(y_1, y_2, \cdots, y_{\alpha_2})$ not on any $S^c_{\alpha_2 - n}$ corresponding to an odd value of n. The countries are accordingly all labelled 0 or 1, and every cycle consisting entirely of 0's or entirely of 1's contains an even number of countries. The map breaks up into a finite number of connected portions, each of which is entirely composed of 0-countries and entirely bounded by 1-countries or entirely composed of 1-countries and entirely bounded by 0-countries. Consider all the 0-countries of a given connected set. They can be reached from an arbitrary 0-country, r_1, by paths which do not pass through vertices and go only through 0-countries. The paths from r_1 to any other country r_2 of this set cross always an odd or always an even number of edges; for if not, on cancelling the common edges of two paths from r_1 to r_2 there would remain at least one odd cycles of 0-countries. Let us color white the country r_1 and all countries of the set connected with it which are reached by crossing an even number of edges and let us color black all countries of the set which are reached by crossing an odd number of edges. No two white countries are adjacent nor are any two black countries. Treat all the connected sets of 0-countries in this fashion. Treat all the connected sets of 1-countries similarly with the colors red and yellow. The

result is a solution of the four color problem. Hence the existence of a point $(y_1, y_2, \cdots, y_{\alpha_2})$ not on any $S^c_{\alpha_2-n}$ for which n is odd implies a solution of the four color problem.

The four color problem has now been reduced to the following form. In a finite projective space of $(\alpha_2 - 1)$ dimensions with three points on a line there are a certain number of spaces $S^c_{\alpha_2-n}$ of dimensionality $\alpha_2 - n$, one for each odd cycle C_n. They all have one point in common (§6). *The map can be colored in four colors if and only if there exists a point not on any of these $S^c_{\alpha_2-n}$'s. There are as many distinct ways of coloring the map (aside from permutations of the colors) as there are real lines in the $(\alpha_2 - 1)$-space which do not meet any $S^c_{\alpha_2-n}$ $(n$, odd).*

11. Another set of equations associated with the map problem arises as follows. If y_i and y_j are two variables which appear in the same one of the equations (3), i. e., which correspond to adjacent regions, let us denote $y_i + y_j$ by a new variable x_k. There will be one x_k for each of the equations (3), i. e., for each edge of the map. The condition that none of the equations (3) be satisfied now takes the form

$$(5) \qquad x_k \neq 0 \quad (k = 1, 2, \cdots, \alpha_1).$$

The set of all x's corresponding to the edges meeting at a vertex of the map is a sum of pairs of y's in which each y appears twice. Hence if the x's meeting at a vertex be $x_a, x_b, x_c, \cdots$, they must satisfy the equation,

$$(6) \qquad x_a + x_b + x_c + \cdots = 0.$$

These equations are evidently the same as (1), §3. In other words, *it is necessary in order to solve the problem to find a solution of the equations* (1) *in which none of the variables vanishes.*

This is also sufficient. For any equation of the form (6) in which $x_a, x_b, \cdots$ are the edges which appear in a cycle is linearly dependent on the equations (1). (It is in fact the sum of the equations (1) corresponding to vertices in one of the two parts into which the surface of the sphere is divided by the cycle.) Hence a solution of the equations (1) is such that the sum of the marks on the edges of any cycle is zero. Suppose this solution labels all the edges of the map with marks different from 0. If now the mark 0 be assigned to an arbitrary region and the other regions be marked according to the rule that the sum of the marks of two adjacent countries shall be equal to the mark of the edge separating them, a unique mark is assigned to each country; otherwise the edges of some cycle would have a sum different from zero. Since the marks of no two adjacent countries are alike, this determines a coloring in four colors.

12. It is well known that the four color problem may be reduced to the problem of coloring a map in which three edges meet at each vertex. In this case the equations (1) discussed in §11 have only three turns each. This form of the equations was found by Mr. A. A. Bennett by a method different from the above. Mr. Bennett however did not regard the variables as marks of a Galois Field.

From the equations (1) in this form can be derived the set of equations modulo three discovered by Heawood.* In any solution of (1) the three values of the variables x_a, x_b, x_c corresponding to the edges which meet at any vertex must be 1, i and i^2 in some order, for this is the only way of satisfying

$$x_a + x_b + x_c = 0$$

by values different from zero. An arbitrary sense on the sphere having been chosen as positive, the three marks on the edges at any vertex follow one another in the positive sense either in the cyclic order 1, i, i^2 or the cyclic order 1, i^2, i. In the first case each mark is obtained from its predecessor by multiplying by i, in the second case by multiplying by i^2. Thus by properly distributing the marks i, and i^2 at the vertices the marks of all edges are determined as soon as the mark on one edge is given. For if the mark α on an edge, e, is given and β is the mark at one of its vertices, v, then the mark on the edge following e in the positive cyclic order at v is $\alpha\beta$ and the mark on the other edge is $\alpha\beta^2$. In order that this process assign a unique mark to each edge, the product of the i's and i^2's multiplied in while describing any closed circuit must be unity.

It is necessary and, in view of the simple connectivity of the sphere, sufficient that this condition be satisfied for the boundary of each country in the map. In view of the identity,

$$i^3 = 1,$$

this means that the sum of the exponents of the i's at the vertices of any country must be divisible by three. Hence if $z_1, z_2, \cdots, z_k$ are the exponents at the vertices of any country they must satisfy the relation,

$$z_1 + z_2 + \cdots + z_k = 0 \qquad \text{(mod. 3)}.$$

The α_2 countries give rise to α_2 equations of this form among α_0 variables representing the vertices of the map. These are Heawood's equations. The matrix of the coefficients of the equations is the matrix analogous to A and B representing the incidence relations of the vertices and countries of the map.

* Quarterly Journal of Pure and Applied Mathematics, vol. 29 (1898), p. 270.

To solve the four color problem it is necessary and sufficient to find a solution of these equations in which none of the variables vanish. The variables may be interpreted as coördinates of points in a finite projective space of α_0-dimensions in which there are four points on every line.

GROUPS WHICH CONTAIN AN ABELIAN SUBGROUP OF PRIME INDEX.

By G. A. Miller.

§ 1. Introduction.

A necessary and sufficient condition that an abelian group contains a subgroup of index p, p being a prime number, is that the order of this group is divisible by p. In what follows we shall therefore assume that the group G under consideration is non-abelian, and we shall use the symbol H to represent an abelian subgroup of index p contained in G. From the fact that every subgroup of prime index in any group is maximal, it results that H is a maximal subgroup of G. If H is non-invariant any two conjugates of H must therefore generate G, and the common operators of these two conjugates must constitute the central of G. In fact, it is evident that the common operators of any two conjugate abelian maximal subgroups* must always constitute the central of a group, but the central is not necessarily contained in two such subgroups.

Every subgroup of index ρ under any group includes $1/\rho'$, $\rho' \gtreqless \rho$, of the operators of every other subgroup of this group. We proceed to prove that $\rho' < \rho$ whenever the two subgroups in question are conjugate and to deduce a few results from this theorem. Suppose that K_1, K_2 are two conjugate subgroups of index n under the group K, and let K_0 represent the subgroup formed by all the common operators of K_1 and K_2. Since the index n' of K_0 under K_1 cannot exceed n, it is only necessary to prove that we arrive at an absurdity by assuming $n' = n$. If n' were equal to n all the operators of K could be represented in the form s_1s_2, where s_1 represents any operator of K_1, and s_2 represents any operator of K_2. It results therefore that all the conjugates of K_1 would be transforms of K_1 under K_2. This is impossible since the operators of K_2 cannot transform K_1 into K_2. It has therefore been proved that $n' < n$. That is, *the index of the subgroup formed by the common operators of two conjugate subgroups, under either of these subgroups, is always smaller than the index of one of these two conjugate subgroups under the entire group.*

From the theorem which has just been quoted it is very easy to derive the known theorem that every subgroup of index p in a group of order p^m

* It should be observed that an abelian maximal subgroup is also a maximal abelian subgroup but the converse is not necessarily true.

is invariant. If such a subgroup were not invariant there would be two conjugate subgroups of this order. The common operators of these subgroups would constitute a subgroup whose index would be less than p under each of these conjugate subgroups. As this index must also be a divisor of p^{m-1} it must be unity. That is, the two subgroups in question must be identical, and hence every subgroup of index p in a group of order p^m is invariant. This theorem is also known to be a special case of the following theorem. In a group of order p^m every non-invariant subgroup is transformed into itself by at least p of its conjugates.*

The theorem in italics includes also the well known theorem that a subgroup of index 2 is necessarily invariant. If there were two conjugate subgroups of this index their common operators would constitute a subgroup whose index under each of these two conjugate subgroups would be less than 2, and hence this index would be unity. It is also evident that two conjugate subgroups of index 3 must always have exactly half their operators in common, and these common operators constitute an invariant subgroup of the entire group.

It results directly from the properties of a direct product that the index of the subgroup formed by the common operators of two invariant subgroups, under either of these subgroups, is *not* always smaller than the index of one of these subgroups under the entire group. In fact, if a group G contains two invariant subgroups G_1, G_2 of the same prime index p, all the common operators of G_1, G_2 constitute an invariant subgroup G_0 of G, since all the common operators of any two invariant subgroups constitute an invariant subgroup. As p divides the orders of all the operators of G which are not contained in G_0, we have proved the theorem: *The common operators of two invariant subgroups of the same prime index p constitute an invariant subgroup of index p under each of these two invariant subgroups.*

§ 2. Groups in which an abelian subgroup of prime index is non-invariant.

Since H is a non-invariant subgroup of index p under G the central of G must be a subgroup of H whose index under H is less than p and $p > 2$. Hence the p conjugates of H are transformed under G according to the metacyclic group of order $p(p-1)$ or according to an invariant subgroup of this group. This transformation group is evidently simply isomorphic with the central quotient group of G, and hence it follows that the index of the central of G under H is a divisor of $p-1$. If i represents this index, the order of the cental quotient group of G is ip, and it is evident that any possible isomorphism between the group of order ip contained in the meta-

* American Journal of Mathematics, vol. 23 (1901), p. 173.

cyclic group of order $p(p-1)$ and any abelian group whatever will result in a group which involves an abelian non-invariant subgroup of index p. It should be observed that direct products are to be included among these possible isomorphisms. We proceed to prove that, conversely, every possible group which involves non-invariant abelian subgroups of index p can be obtained by establishing such an isomorphism.

To prove this theorem we may first observe that each of the Sylow subgroups of G is abelian and that the independent generators of the Sylow subgroup of order p^m may be so selected that all except one of them are in the central of G. The one which is not in this central is of order p and generates the commutator subgroup of G. In fact, if s is any operator of G which is not in the abelian group generated by the central of G and the Sylow subgroup of order p^m, it follows that s transforms a cyclic subgroup of the Sylow subgroup of order p^m into itself, since the number of the cyclic subgroups of order p^a, in any group of order p^m, which do not have their generators in a given non-cyclic subgroup of order p^{m-1} is always a multiple of p, $p > 2$.* Since the order of s is prime to p and since the pth powers of the generators of the cyclic subgroup of order p^a which s transforms into itself are invariant, it results that $\alpha = 1$ and that this cyclic subgroup is the commutator subgroup of G, as was observed above.

From the preceding paragraph it follows that whenever the order of G is divisible by p^m, $m > 1$, G must be a direct product of an abelian group of order p^{m-1} and of some other factor group. It also results that all the Sylow subgroups of G whose orders are prime to ip, if such Sylow subgroups exist in G, are *proper factors* of G: that is, G is the direct product of these factor groups and of some other subgroup. Hence it results that *G may be constructed by establishing an isomorphism between an abelian group and the group of order ip contained in the metacyclic group of order $p(p-1)$, whenever G contains a non-invariant abelian subgroup of index p.*

It has been observed that G is divisible whenever any one of its Sylow subgroups is contained entirely in its central, and also when the order of G is divisible by p^2. Suppose that G does not satisfy either of these two conditions. Hence G may be obtained by establishing an isomorphism between an abelian group G_1, whose order does not involve any factor which is prime to i, and the invariant subgroup G_2 of order ip contained in the metacyclic group of order $p(p-1)$. It is also clear that G_1 is cyclic whenever G is indivisible since the Sylow subgroups of G_2 are cyclic. Moreover, G cannot be indivisible when G involves the direct product of any Sylow subgroup of G_1 and a Sylow subgroup of G_2, or when the order of a Sylow subgroup of G_1 is equal to or less than the order of the corresponding

* Proceedings of the London Mathematical Society, series 2, vol. 2 (1905), p. 142.

Sylow subgroup of G_2 and the isomorphism between these Sylow subgroups is α, β, where $\alpha > 1$. Hence we are led to the following theorem: *In order that an indivisible group G involves a non-invariant subgroup of index p, it is necessary and sufficient that the following three conditions are satisfied: G may be constructed by establishing an isomorphism, between a group G_2 of order ip, $i > 1$, contained in the metacyclic group of order $p(p-1)$ and a cyclic group G_1 whose order involves no prime factors except those of i; in this isomorphism there is a $(1, \alpha)$ correspondence between the operators of two Sylow subgroups contained in G_1 and G_2 respectively, whenever the order of the Sylow subgroup of G_1 does not exceed the order of the corresponding Sylow subgroup of G_2; the isomorphism between two Sylow subgroups of G_1 and G_2 respectively is not a direct product unless the former of these subgroups is the identity.*

§ 3. Groups in which an abelian subgroup of prime index is invariant.

It results from the preceding section that G cannot involve both an invariant and also a non-invariant abelian subgroup of the same prime index. From the theorem proved at the end of the Introduction it follows that the non-abelian group G contains exactly $p + 1$ abelian subgroups of prime index p whenever it contains more than one such subgroup. When G contains $p + 1$ such subgroups the central quotient group of G is the non-cyclic group of order p^2, and the common operators of these $p + 1$ invariant abelian subgroups constitute the central of G. As H is the direct product of its Sylow subgroups, *G is the direct product of an abelian group whose order is prime to p and of a non-abelian group of order p^m, $m > 2$, whenever G contains more than one invariant abelian subgroup of index p. This non-abelian group of order p^m contains exactly $p + 1$ abelian subgroups of order p^{m-1}.*

As every direct product of an abelian group and of a non-abelian group of order p^m which contains more than one abelian subgroup of order p^{m-1}, evidently constitutes a non-abelian group involving more than one abelian subgroup of index p, it results that a necessary and sufficient condition that a non-abelian group contains more than one abelian invariant subgroup of index p is that it satisfies the conditions imposed on G in the theorem of the preceding paragraph. It also results from this theorem and from those of the preceding section that every non-abelian group which contains more than one abelian subgroup of index p has a commutator subgroup of order p, and these abelian subgroups are invariant or non-invariant as the commutators of order p are invariant or non-invariant under the entire group. As a very elementary illustration of this theorem it may be observed that the dihedral group of order $2p$ contains non-invariant commutators of order p and hence its abelian subgroups of index p are non-invariant.

In what follows we shall assume that H is cyclic and we shall first consider the case when $p = 2$. It will be convenient to assume that the order h of H is written in the form

$$h = 2^{\alpha_0}p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_\lambda^{\alpha_\lambda},$$

where $p_1, p_2, \cdots, p_\lambda$ are distinct odd prime numbers and all the exponents $\alpha_1, \alpha_2, \cdots, \alpha_\lambda$ exceed 0. Since the group of isomorphisms of the cyclic group of order p^m, $p > 2$, involves one and only one operator of order 2 it results that the number of possible non-abelian G's, when $\alpha_0 = 0$, is given by the formula

$$\lambda + \frac{\lambda(\lambda-1)}{2!} + \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \cdots + \lambda + 1 = (1+1)^\lambda - 1 = 2^\lambda - 1.$$

It may be observed that the number of these groups depends only on λ and not upon the values of $\alpha_1, \alpha_2, \cdots, \alpha_\lambda$. The formula follows directly from the fact that the operator of order 2 which must be adjoined to H to generate G may transform the Sylow subgroups of H in as many different ways as the integer expressed by this formula.

When $\alpha_0 = 1$, H involves an operator of order 2 which is invariant under G and may be either dihedral or dicyclic. There are therefore just twice as many possible groups in this case as in the case which has just been considered. That is, the number of possible non-abelian groups when $\alpha_0 = 1$ is given by the formula $2^{\lambda+1} - 2$. When $\alpha_0 = 2$ the group of isomorphism of the Sylow subgroup of order 4 in H involves also one operator of order 2, and hence the number of possible non-abelian groups in this case is $2^{\lambda+2} - 2$.

It remains to consider the case when $\alpha_0 \gtreqless 3$. In this case we have to add to the groups just considered those which result when the operators of the Sylow subgroup of order 2^{α_0} in H are transformed either into their $2^{\alpha_0-1} - 1$ or into their $2^{\alpha_0-1} + 1$ powers. That is, we have to add twice the number of the possible groups when $\alpha_0 = 0$, including the case when the group is abelian. Hence the possible number of non-abelian groups in this case is given by the formula

$$2^{\lambda+2} - 2 + 2^{\lambda+1}.$$

As the number of the possible abelian groups which contain a cyclic subgroup of order h is 2 when h is even and only one when h is odd we have established the following theorem, which includes abelian groups: *If λ represents the number of the distinct odd prime factors of the order h of a cyclic group H, there are exactly 2^λ groups of order $2h$ which include H whenever h is an odd number, and there are exactly $2^{\lambda+1}$ such groups when h is the double of an odd number. When h is divisible by 4 but not by 8 the number of these*

possible groups is $2^{\lambda+2}$, *and this number is* $3.2^{\lambda+1}$ *when h is divisible by* 8. Although this theorem was developed under the assumption that $\lambda > 0$ it is clearly true also when $\lambda = 0$. It may be observed that the number of these distinct groups increases rapidly with the number of the distinct prime factors of the order of the given cyclic group. For instance, there are 24 distinct groups of order 240 which include a common cyclic subgroup of order 120.

When the prime index p of the invariant cyclic subgroup H contained in the non-abelian group G is odd, it follows that either p^2 divides h or that p divides at least one of the numbers $p_1 - 1$, $p_2 - 1$, $\cdots$, $p_\lambda - 1$. If p^2 divides h, we let $\gamma - 1$ represent the number of the numbers $p_1 - 1$, $p_2 - 1$, $\cdots$, $p_\lambda - 1$ that are divisible by p, and, if p^2 does not divide h, we let γ represent this number. As the group of isomorphisms of the cyclic group of order p^m is cyclic and of order $p^{m-1}(p-1)$, while the group of isomorphisms of the cyclic group of order 2^m is of order 2^{m-1}, it is easy to see that the number of the distinct possible non-abelian groups of order hp which include H is given by the following formula:

$$\gamma + \frac{1}{2!}(p-1)\gamma(\gamma-1) + \frac{1}{3!}(p-1)^2\gamma(\gamma-1)(\gamma-2) + \cdots + (p-1)^{\gamma-1}.$$

If we combine this formula with the theorem of the preceding paragraph and observe that the number of abelian groups of order hp, which include H, is 2 or 1 according as h is or is not divisible by p, we obtain the total number of the groups of order hp which include the cyclic group H as an invariant subgroup of prime index.

ON INFINITE SYSTEMS OF LINEAR INTEGRAL EQUATIONS.

By Louis Brand.

The principal problem of this paper is the solution of a denumerably infinite system of linear integral equations of the type

$$\int_a^b \varphi_i(x)\xi(x)dx = b_i \qquad (i = 1, 2, \cdots),$$

the φ's being given functions and the b's given constants. The solution of such a system of equations, besides its intrinsic importance, enables us to approach in a simple manner other matters such as the existence of biorthogonal systems of functions and the expansion of an arbitrary function in terms of a given system of functions by the method of least squares. The method of procedure is closely analogous to that employed in an earlier paper in these *Annals** by Professor Bôcher and myself, treating of the solution of an infinite number of linear equations in an infinite number of unknowns, and is chiefly characterized by the preliminary solution of a finite number of equations and the treatment of the infinite system as a limiting case. The trend of this analogy is fully exhibited in Theorems 1 and 2; subsequently the proofs of the theorems having close analogues in (I) are given in much less detail—sometimes in mere outline when reference to this article will make the detailed procedure evident.

This paper is a development of a manuscript handed me by Professor Bôcher a year ago in which the method of proceeding for §§ 2, 3, and the beginning of § 6, was sketched in outline. This manuscript was written in June, 1910, several months before the appearance of a paper by F. Riesz† in which it was mentioned how Schmidt's method,‡ of which a modification is used in (I), can be applied to the problem here considered. The method here developed has in comparison to that suggested by Riesz§ all the advantages which the treatment in (I) has over the original treatment of that problem by Schmidt.

1. We will consider real or complex functions of the real variable x on the arbitrary interval $a \leqq x \leqq b$, denoting such functions by f, φ, etc.,

* June, 1912. This article will be referred to as (I).

† "Systeme integrierbare Funktionen," Mathematische Annalen, 69 (1910), p. 451, p. 469.

‡ "Über die Auflösung linearer Gleichungen mit unendlich vielen Unbekannten," Rend. Circ. Matem. Palermo, 25 (1908), pp. 53–77.

§ The method actually used by Riesz in the article above cited differs from both of these and leads to different results.

and their conjugate imaginary functions by $\bar{f}$, $\bar{\varphi}$, etc., omitting the arguments. All integrals shall be taken in the sense of Lebesgue, and the term *integrable* shall mean integrable in this sense* on the interval (a, b). Ω is the class of all such functions φ for which $|\varphi|^2$ is integrable. It may be readily shown that if $|\varphi|^2$ is integrable (we denote functions of the class Ω by Greek letters) both $|\varphi|$ and φ are integrable.† But the chief characteristic of functions of the class Ω is expressed in the following theorem: *The product of any two functions of the class Ω is integrable; and every function which yields an integrable product with all the functions of the class Ω is itself a member of this class.*‡ From this theorem it follows that the sum of two functions of the class Ω is likewise a member of this class; or, more generally, any linear combination of functions of the class Ω belongs to this class.

A function is said to be *essentially zero* if it differs from zero merely on a point set of content§ zero. The integral of a function essentially zero is, of course, zero. The n functions $f_1, \cdots, f_n$ are said to be *essentially linearly dependent* in (a, b) if there exist n constants, $c_1, \cdots, c_n$, not all zero, such that $c_1 f_1 + \cdots + c_n f_n$ is essentially zero in this interval.

By the *norm* of a function φ of the class Ω is understood the real constant

$$\text{norm } \varphi = \int_a^b \varphi\bar{\varphi}dx = \int_a^b |\varphi|^2 dx.$$

Norm $\varphi = 0$ when and only when φ is essentially zero, and is otherwise positive.

In the following we shall omit the limits of integration, a, b, and the dx in writing integrals; these are to be understood in all cases.

2. Consider now the system of n homogeneous integral equations

$$\int \varphi_1\xi = 0, \qquad \int \varphi_2\xi = 0, \qquad \cdots, \qquad \int \varphi_n\xi = 0, \tag{1}$$

where $\varphi_1, \varphi_2, \cdots, \varphi_n$ are functions of the class Ω. We propose to determine all the functions ξ of the class Ω satisfying (1).

THEOREM 1. *If ξ is a solution of (1) which is essentially linearly dependent upon $\bar{\varphi}_1, \bar{\varphi}_2, \cdots, \bar{\varphi}_n$, then ξ is essentially zero.*

Suppose that

$$\xi = c_1\bar{\varphi}_1 + c_2\bar{\varphi}_2 + \cdots + c_n\bar{\varphi}_n$$

* Lebesgue, Leçons sur l'intégration, Paris (1904), p. 115. Lebesgue terms functions integrable in this sense *sommable*. See also F. Riesz, l. c., § 1.

† Cf. F. Riesz, l. c., § 2.

‡ Lebesgue, "Sur les intégrales singulières," Annales de la Faculté des Sciences de Toulouse, 3° serie, t. I, p. 37, p. 39; F. Riesz, l. c., p. 459. Riesz proves a more general theorem than the one stated above.

§ Content as defined by Lebesgue, Leçons sur l'intégration, p. 102 et seq., i. e., *mesure*.

(essentially). Multiplying equations (1) by $\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_n$ respectively and adding we get

$$\int (\bar{c}_1\varphi_1 + \bar{c}_2\varphi_2 + \cdots + \bar{c}_n\varphi_n)\xi = \int \xi\bar{\xi} = \int |\xi|^2 = 0.$$

Hence ξ is essentially zero, as was to be proved.

We are now in position to obtain a criterion for the essential linear dependence of n functions of the class Ω. If $\varphi_1, \varphi_2, \cdots, \varphi_n$ are essentially linearly dependent

$$c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n = 0$$

(essentially), where not all of the c's are zero. Multiplying this relation in succession by $\bar{\varphi}_1, \bar{\varphi}_2, \cdots, \bar{\varphi}_n$, and integrating from a to b we get the n equations

$$c_1\int \varphi_1\bar{\varphi}_i + c_2\int \varphi_2\bar{\varphi}_i + \cdots + c_n\int \varphi_n\bar{\varphi}_i = 0 \qquad (i = 1, 2, \cdots, n).$$

Since the c's are not all zero the determinant of this system must vanish; thus, interchanging rows and columns in this determinant, we have

$$(2) \qquad \begin{vmatrix} \int \varphi_1\bar{\varphi}_1 & \int \varphi_1\bar{\varphi}_2 & \cdots & \int \varphi_1\bar{\varphi}_n \\ \int \varphi_2\bar{\varphi}_1 & \int \varphi_2\bar{\varphi}_2 & \cdots & \int \varphi_2\bar{\varphi}_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int \varphi_n\bar{\varphi}_1 & \int \varphi_n\bar{\varphi}_2 & \cdots & \int \varphi_n\bar{\varphi}_n \end{vmatrix} = 0.$$

This relation is therefore a necessary condition for essential linear dependence.

It is also a sufficient condition. For suppose that (2) is fulfilled; then the n sets of constants forming the rows of the determinant are linearly dependent, and we have

$$\int \bar{\varphi}_i(c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n) = 0 \quad (i = 1, 2, \cdots, n),$$

where not all of the c's vanish. From Theorem 1 we now readily infer that

$$c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n = 0$$

(essentially), which establishes the essential linear dependence of $\varphi_1, \varphi_2, \cdots, \varphi_n$.

We term the determinant in (2) the *Gramian* of $\varphi_1, \varphi_2, \cdots, \varphi_n$ and denote it by $G(\varphi_1, \varphi_2, \cdots, \varphi_n)$. Thus we have

THEOREM 2. *A necessary and sufficient condition that the n functions*

*$\varphi_1, \varphi_2, \cdots, \varphi_n$ of the class Ω be essentially linearly dependent is that their Gramian vanish.**

We turn now to the solution of the system (1), assuming that $\varphi_1, \varphi_2, \cdots, \varphi_n$ are essentially linearly independent, so that $G(\varphi_1, \cdots, \varphi_n) \neq 0$. This entails no loss in generality; since if, for example, φ_n were essentially linearly dependent on $\varphi_1, \cdots, \varphi_{n-1}$, the equation $\int \varphi_n \xi = 0$ would be satisfied by all the solutions of the preceding system and could therefore be omitted. Every solution ξ_1 of (1) of the class Ω may be written in the form

$$\xi_1 = c_1\bar{\varphi}_1 + \cdots + c_n\bar{\varphi}_n + \bar{\eta}, \tag{3}$$

η being some function of the class Ω. In order that this be a solution the constants c_i must satisfy the system of linear equations

$$c_1\int \varphi_i\bar{\varphi}_1 + \cdots + c_n\int \varphi_i\bar{\varphi}_n + \int \varphi_i\bar{\eta} = 0 \quad (i = 1, 2, \cdots, n),$$

which may be solved by Cramer's rule as their determinant is precisely $G(\varphi_1, \cdots, \varphi_n)$. These values substituted in (3) give

$$\xi_1 = \frac{\begin{vmatrix} \int \varphi_1\bar{\varphi}_1 & \cdots & \int \varphi_1\bar{\varphi}_n & \int \varphi_1\eta \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int \varphi_n\bar{\varphi}_1 & \cdots & \int \varphi_n\bar{\varphi}_n & \int \varphi_n\bar{\eta} \\ \bar{\varphi}_1 & \cdots & \bar{\varphi}_n & \bar{\eta} \end{vmatrix}}{G(\varphi_1, \cdots, \varphi_n)}. \tag{4}$$

That, no matter what the function η of the class Ω may be, the formula (4) always gives a solution of (1) belonging to this class is seen by direct substitution.

THEOREM 3. *If in equations (1) $\varphi_1, \varphi_2, \cdots, \varphi_n$ are essentially linearly independent functions belonging to the class Ω, their general solution of this class is given by (4), η being an arbitrary function of the class Ω.*†

A line of reasoning analogous to that on p. 171 of (I) shows that two different $\bar{\eta}$'s yield essentially the same ξ_1 when and only when their difference is essentially linearly dependent upon $\bar{\varphi}_1, \bar{\varphi}_2, \cdots, \bar{\varphi}_n$.

To find a formula for norm ξ_1 we form $\int \xi_1\bar{\xi}_1$ from (3), whence, remembering that the norm is real,

$$\text{norm } \xi_1 = \int \eta\xi_1; \tag{5}$$

---

* We note in passing that $\varphi_1, \varphi_2, \cdots, \varphi_n$ are connected by the same linear relation that connects the rows of their Gramian, *written as above*.

† If η is given the solution ξ_1 is essentially unique.

and from (4) the latter integral is seen to be equal to

$$\text{(6)} \qquad \text{norm}\ \xi_1 = \frac{G(\varphi_1, \cdots, \varphi_n, \eta)}{G(\varphi_1, \cdots, \varphi_n)}.$$

Just as in (I), p. 171, this relation may be used to establish

THEOREM 4. *The Gramian of any number of essentially linearly independent functions of the class Ω is real and positive.*

3. We now pass to the non-homogeneous system

$$\text{(7)} \qquad \int \varphi_1\xi = b_1, \qquad \int \varphi_2\xi = b_2, \qquad \cdots, \qquad \int \varphi_n\xi = b_n,$$

where we again assume that $\varphi_1, \cdots, \varphi_n$ are essentially linearly independent functions of the class Ω. We seek a solution of the form

$$\text{(8)} \qquad \xi_0 = c_1\bar{\varphi}_1 + \cdots + c_n\bar{\varphi}_n.$$

Substituting this in (7) we obtain n linear equations to determine the c's. These can, as above, be solved by Cramer's rule; and the result substituted in (8) gives

$$\text{(9)} \qquad \xi_0 = \frac{\begin{vmatrix} \int \varphi_1\bar{\varphi}_1 & \cdots & \int \varphi_1\bar{\varphi}_n & -b_1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int \varphi_n\bar{\varphi}_1 & \cdots & \int \varphi_n\bar{\varphi}_n & -b_n \\ \bar{\varphi}_1 & \cdots & \bar{\varphi}_n & 0 \end{vmatrix}}{G(\varphi_1, \cdots, \varphi_n)}.$$

That this is really a solution of (7) we see by direct substitution. Hence we may state

THEOREM 5. *If $\varphi_1, \varphi_2, \cdots, \varphi_n$ are essentially linearly independent functions of the class Ω, the equations* (7) *have one and only one solution of the form* (8), *and this is given by* (9).

The general solution of (7) is obtained by adding to the particular solution (9) the general solution (4) of the homogeneous system (1); it is therefore

$$\text{(10)} \qquad \xi = \xi_0 + \xi_1 = \frac{\begin{vmatrix} \int \varphi_1\bar{\varphi}_1 & \cdots & \int \varphi_1\bar{\varphi}_n & \int \varphi_1\bar{\eta} - b_1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int \varphi_n\bar{\varphi}_1 & \cdots & \int \varphi_n\bar{\varphi}_n & \int \varphi_n\eta - b_n \\ \bar{\varphi}_1 & \cdots & \bar{\varphi}_n & \bar{\eta} \end{vmatrix}}{G(\varphi_1, \cdots, \varphi_n)}.$$

The solution (9) of (7), which is characterized by being the only solution of (7) which is linearly dependent upon the $\bar{\varphi}$'s, shall be called the *principal*

solution of (7). Another characteristic property of ξ_0, which can be readily deduced from (10) and (8), is that

$$\text{norm } \xi = \text{norm } \xi_0 + \text{norm } \xi_1 \tag{11}$$

and hence

$$\text{norm } \xi \geqq \text{norm } \xi_0,$$

the equality sign holding only when ξ_1 is essentially zero, in which case ξ and ξ_0 are essentially equal. Thus we have

THEOREM 6. *Among the solutions of* (7) *no others have so small a norm as the principal solution except those essentially equal to it.*

To obtain a formula for norm ξ_0 we form $\int \xi_0\bar{\xi}_0$ from (9):*

$$\text{norm } \xi_0 = -\frac{\begin{vmatrix} \int \varphi_1\bar{\varphi}_1 & \cdots & \int \varphi_1\bar{\varphi}_n & b_1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int \varphi_n\bar{\varphi}_1 & \cdots & \int \varphi_n\bar{\varphi}_n & b_n \\ \bar{b}_1 & \cdots & \bar{b}_n & 0 \end{vmatrix}}{G(\varphi_1, \cdots, \varphi_n)}. \tag{12}$$

Norm ξ is now given by (11).

Let us now write those particular c_i's which when substituted in (8) give ξ_0 as $c_i{}^0$, so that

$$\xi_0 = c_1{}^0\bar{\varphi}_1 + \cdots + c_n{}^0\bar{\varphi}_n.$$

Then

$$\text{norm } \xi_0 = \int \xi_0\bar{\xi}_0 = \sum_{i=1}^{n} c_i{}^0\bar{b}_i. \tag{13}$$

We now inquire as to what special role the solution ξ_0 plays in the totality of expressions essentially of the form $\Sigma_{i=1}^n c_i\bar{\varphi}_i$, regarded as functions of the c's, which may vary subject to the condition

$$\text{norm} \sum_{i=1}^{n} c_i\bar{\varphi}_i = \sum_{i=1}^{n} c_i\bar{b}_i \tag{14}$$

suggested by (13). For a convenience in the following argument we will exclude the case $c_1 = c_2 = \cdots = c_n = 0$; then, as the φ's are essentially linearly independent, (14) requires, in particular, that $\Sigma_{i=1}^n c_i\bar{b}_i$ be always real and positive. Letting ξ represent an arbitrary solution of (7), we have from Theorems 2 and 4

* As in (I), p. 173, footnote, we may use (12) to prove the THEOREM: *If the Gramian of essentially linearly independent functions of the class Ω is bordered by constants, which do not all vanish, so as to form a bordered Gramian of the type of that in* (12), *this bordered Gramian is negative.*

$$G(\xi, \Sigma c_i\bar{\varphi}_i) = \begin{vmatrix} \int |\xi|^2 & \int \xi\Sigma\bar{c}_i\varphi_i \\ \int \bar{\xi}\Sigma c_i\bar{\varphi}_i & \int |\Sigma c_i\bar{\varphi}_i|^2 \end{vmatrix} \geqq 0,$$

or

$$\int |\xi|^2 \cdot \int |\Sigma c_1\bar{\varphi}_i|^2 \geqq [\Sigma c_i\bar{b}_i]^2.$$

In view of (14) this yields

(15) $$\text{norm } \xi \geqq \Sigma c_i\bar{b}_i,$$

the sign of equality holding when and only when ξ and $\Sigma c_i\bar{\varphi}_i$ are essentially linearly dependent. This is clearly not the case when ξ is essentially *not* of the form $\Sigma c_i\bar{\varphi}_i$, i. e., when ξ is essentially different from ξ_0; hence writing $c_i = c_i^0$ in (15), we have that norm $\xi >$ norm ξ_0 except when ξ and ξ_0 are essentially equal. This constitutes another proof of Theorem 6.

Again, letting $\xi = \xi_0$ in (15), we have

$$\text{norm } \xi_0 \geqq \Sigma c\bar{b}_i,$$

the equality sign holding only when

$$\xi_0 = k\Sigma c_i\bar{\varphi}_i$$

(essentially), in which case we find from (14)

$$\text{norm } \xi_0 = |k|^2\Sigma c_i\bar{b}_i = k\Sigma c_i\bar{b}_i.$$

Hence $k = 1$, the value $k = 0$ being excluded as ξ_0 is not essentially zero, and

$$\Sigma(c_i^0 - c_i)\bar{\varphi}_i = 0$$

(essentially), which, in view of essential linear independence of the $\bar{\varphi}$'s, can only be true when $c_i^0 = c_i$. Thus we have shown that the polynomial $\Sigma c_i\bar{b}_i$ in $c_1, \cdots, c_n$, when subject to the condition (14) attains its maximum value, norm ξ_0, when and only when $c_i = c_i^0$.

THEOREM 7. *In the principal solution*

$$\xi_0 = c_1^0\bar{\varphi}_1 + \cdots + c_n^0\bar{\varphi}_n$$

the constants $(c_1^0, \cdots, c_n^0)$ *form that set of values which render the polynomial* $c_1\bar{b}_1 + \cdots + c_n\bar{b}_n$ *a maximum when it is subject to the condition* (14).

Thus the finding of the principal solution of equations (7) may be regarded as a problem in conditional maxima, and in fact ξ_0 may be obtained by the classical procedure for such problems.*

4. Before giving an application of the preceding results we lay down the following definitions:

* See F. Riesz, l. c., § 9, where a more general problem is treated in this way.

A function of the class Ω is said to be *normalized* when its norm is unity.

Two functions φ, ψ are said to be *orthogonal* in the interval $a \leqq x \leqq b$ if $\int_a^b \varphi\bar{\psi}dx$, and hence also $\int_a^b \bar{\varphi}\psi dx$, is zero. The system of functions $\{\varphi_i\}$, finite or infinite in number, is termed orthogonal in the interval $a \leqq x \leqq b$ when

$$\int_a^b \varphi_i\bar{\varphi}_j dx = \int_a^b \bar{\varphi}_i\varphi_j dx = 0 \qquad (i \neq j).$$

We now propose to solve the problem: *Given the system of functions* $\{\varphi_n\}$ *of the class* Ω, *none of which are essentially linearly dependent, to find a system of normalized, orthogonal functions* $\{\Phi_n\}$ *such that*

$$\Phi_n = \sum_{i=1}^{i=n} a_{ni}\varphi_i, \qquad a_{nn} \neq 0,$$

the a's being constants.

Φ_n must satisfy the n equations

$$\int \bar{\Phi}_i\xi = \begin{cases} 0 & (i \neq n), \\ 1 & (i = n); \end{cases}$$

and upon substituting for $\bar{\Phi}_i$ the linear expression in the $\bar{\varphi}$'s an evident reduction yields the equivalent system

$$\int \bar{\varphi}_1\xi = 0, \qquad \int \bar{\varphi}_2\xi = 0, \qquad \cdots, \qquad \int \bar{\varphi}_{n-1}\xi = 0, \qquad \int \bar{\varphi}_n\xi = \frac{1}{\bar{a}_{nn}}.$$

From Theorem 5 we see that the only solution of this system having the form of Φ_n is its principal solution ξ_0. To determine a_{nn} we have from (12)

$$\text{norm } \xi_0 = \frac{1}{|a_{nn}|^2} \cdot \frac{G_{n-1}}{G_n} = 1,$$

$$(16) \qquad |a_{nn}| = \sqrt{\frac{G_{n-1}}{G_n}},$$

where we have written $G_n = G(\varphi_1, \cdots, \varphi_n) = G(\bar{\varphi}_1, \cdots, \bar{\varphi}_n)$ with the convention that $G_0 = 1$. Thus, choosing a_{nn} in accordance with (16), we may write $\Phi_n = \xi_0$. However Φ_n is not uniquely determined for there are infinitely many complex numbers satisfying (16); but if we impose the further condition that a_{nn} be real and positive $a_{nn} = \sqrt{G_{n-1}/G_n}$ and Φ_n will be uniquely determined. Then we have from (9), after interchanging rows and columns in the determinant in the numerator,*

* These formulæ are given by Kowalewski, Einführung in die Determinantentheorie, Leipzig (1909), p. 337.

$$(17)\qquad \Phi_n = \frac{1}{\sqrt{G_{n-1}G_n}}\begin{vmatrix} \int \varphi_1\bar{\varphi}_1 & \cdots & \int \varphi_1\bar{\varphi}_{n-1} & \varphi_1 \\ \int \varphi_2\bar{\varphi}_1 & \cdots & \int \varphi_2\bar{\varphi}_{n-1} & \varphi_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int \varphi_n\bar{\varphi}_1 & \cdots & \int \varphi_n\bar{\varphi}_{n-1} & \varphi_n \end{vmatrix} \qquad (n = 1, 2, \cdots).$$

Denoting the cofactor of the element in the last column and ith row of G_n by $G_n^{(i)}$, it is clear that

$$(18)\qquad a_{ni} = \frac{G_n^{(i)}}{\sqrt{G_{n-1}G_n}} \qquad (i = 1, 2, \cdots, n).$$

Thus the problem is solved; if we require that a_{nn} be real and positive there is only one solution and this is given by (17).

5.* We proceed to establish two important inequalities. If φ and ψ are functions of the class Ω we have from Theorems 2 and 4 that $G(\varphi, \bar{\psi}) \geqq 0$, which upon expansion gives Schwarz's Inequality:

$$(19)\qquad \left|\int \varphi\psi\right| \leqq \left[\int |\varphi|^2\right]^{\frac{1}{2}}\left[\int |\psi|^2\right]^{\frac{1}{2}}.$$

Applying this result to $\int (\varphi + \psi)(\bar{\varphi} + \bar{\psi})$ we deduce further that

$$(20)\qquad \left[\int |\varphi + \psi|^2\right]^{\frac{1}{2}} \leqq \left[\int |\varphi|^2\right]^{\frac{1}{2}} + \left[\int |\psi|^2\right]^{\frac{1}{2}}.$$

We next lay down the following

DEFINITION. *The sequence of functions* $\{\varphi_n\}$ *of the class* Ω *is said to converge in the mean to the function* φ *of this class if*

$$\lim_{n=\infty} \int |\varphi - \varphi_n|^2 = 0,\dagger$$

and we write $\lim\limits_{n=\infty} \varphi_n = \varphi$.

The function φ is essentially unique; that is, if $\{\varphi_n\}$ converges in the mean to both φ and ψ, these functions are essentially equal. For by (20)

$$\left[\int |\varphi - \psi|^2\right]^{\frac{1}{2}} = \left[\int |\varphi - \varphi_n + \varphi_n - \psi|^2\right]^{\frac{1}{2}} \leqq \left[\int |\varphi - \varphi_n|^2\right]^{\frac{1}{2}} + \left[\int |\varphi_n - \psi|^2\right]^{\frac{1}{2}},$$

and upon passing to the limit $n = \infty$ we have $\int |\varphi - \psi|^2 = 0$.

It is clear that uniform convergence implies mean convergence. On

* This article is closely analogous to § 5 of (I), to which reference should be made for details of proofs.

† E. Fischer, "Sur la convergence en moyenne," Comptes Rendus, May, 1907, p. 1022. For a generalization of this conception see F. Riesz, l. c., p. 464.

We note also that if the sequence $\{\varphi_n\}$ of the class Ω fulfils this relation, φ is necessarily of the class Ω. For the definition implies that $\varphi - \varphi_n$ is of the class Ω for sufficiently large values of n, and hence the sum of $\varphi - \varphi_n$ and φ_n is also of this class.

the contrary, mean convergence does not even imply convergence in the ordinary sense.

If $\lim\limits_{n=\infty} \varphi_n = \varphi$, $\lim\limits_{n=\infty} \psi_n = \psi$, it follows from (20) that

(21) $$\lim_{n=\infty} (\varphi_n + \psi_n) = \varphi + \psi.$$

Furthermore

(22) $$\lim_{n=\infty} \int \varphi_n \psi_n = \varphi\psi$$

as we readily prove upon writing

$$\int (\varphi\psi - \varphi_n\psi_n) = \int (\varphi - \varphi_n)\psi + \int (\psi - \psi_n)\varphi - \int (\varphi - \varphi_n)(\psi - \psi_n)$$

and applying (19). Important special cases of (22) are

(23) $$\lim_{n=\infty} \int \varphi_n\psi = \int \varphi\psi;$$

(24) $$\lim_{n=\infty} \text{norm } \varphi_n = \text{norm } \varphi.$$

We now state without proof an important criterion for mean convergence due to E. Fischer.*

THEOREM 8. *A necessary and sufficient condition that the sequence of functions* $\{\varphi_n\}$ *of the class* Ω *converge in the mean to a function of this class is that to every positive* ϵ *there correspond an integer* N *such that*

$$\int |\varphi_n - \varphi_m|^2 < \epsilon, \qquad m, n > N.\dagger$$

The infinite series of functions of the class Ω, $\varphi_1 + \varphi_2 + \cdots$, is said to converge in the mean to the function σ when the sum of its first n terms, σ_n, converges in the mean to σ. From Theorem 8 we see that a necessary and sufficient condition for the mean convergence of the above series is that, to every positive ϵ, there correspond an integer N such that

(25) $$\int |\sigma_n - \sigma_m|^2 = \int |\varphi_{m+1} + \varphi_{m+2} + \cdots + \varphi_n|^2 < \epsilon, \qquad m, n > N.$$

If the φ's are all mutually orthogonal (25) becomes

$$\int |\varphi_{m+1}|^2 + \int |\varphi_{m+2}|^2 + \cdots + \int |\varphi_n|^2 < \epsilon, \qquad m, n > N;$$

and as this is precisely a necessary and sufficient condition that the series $\int |\varphi_1|^2 + \int |\varphi_2|^2 + \cdots$ converge, we have proved

* Sur la convergence en moyenne, l. c.

† The apparently redundant phrase "to a function of the class Ω" is inserted because E. Fischer *defines* the sequence to be convergent in the mean when this relation holds and then establishes the existence of a function of the class Ω to which the sequence converges in the mean.

For other proofs of this theorem see Weyl, "Über die Konvergenz von Reihen die nach Orthogonalfunktionen fortschreiten," Mathematische Annalen, 67 (1909), p. 243; Plancherel, "Contribution à l'étude de la représentation d'une fonction arbitraire," etc., Rend. Circ. Matem. Palermo, 30 (1910), p. 292. See also F. Riesz, l. c., p. 468, where a generalization of this theorem is proved.

THEOREM 9. *A series of mutually orthogonal functions of the class* Ω *converges in the mean to a function* σ *of this class when and only when the series of their norms converges.**

Moreover as

$$\int |\sigma_n|^2 = \int |\varphi_1|^2 + \int |\varphi_2|^2 + \cdots + \int |\varphi_n|^2$$

we infer from (24) the

COROLLARY. *If the conditions of Theorem* 9 *are fulfilled, the norm of the function* σ *is equal to the series of the norms of the terms.*

6. We are now in position to consider the infinite system of homogeneous equations

$$\int \varphi_1\xi = 0, \qquad \int \varphi_2\xi = 0, \qquad \cdots, \tag{26}$$

where we assume that $\varphi_1, \varphi_2, \cdots$ are all of the class Ω and none of them are essentially linearly dependent. The general solution of the first n of these equations, which we denote by $\xi_1^{(n)}$, is given by formula (4). We now express $\xi_1^{(m)}$ as the sum of m terms of a series, writing

$$\xi_1^{(m)} = \xi_1^{(1)} + \sum_{n=2}^{n=m} (\xi_1^{(n)} - \xi_1^{(n-1)});$$

and by means of a device entirely analogous to that used in (I), pp. 179, 180, we find that†

$$\xi_1^{(m)} = \bar{\eta} - \sum_{n=1}^{n=m} \frac{H_n}{G_{n-1}G_n}\psi_n, \tag{27}$$

where we have written

$$\psi_1 = \bar{\varphi}_1, \qquad \psi_n = \begin{vmatrix} \int \varphi_1\bar{\varphi}_1 & \cdots & \int \varphi_1\bar{\varphi}_n \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \int \varphi_{n-1}\bar{\varphi}_1 & \cdots & \int \varphi_{n-1}\bar{\varphi}_n \\ \bar{\varphi}_1 & \cdots & \bar{\varphi}_n \end{vmatrix} \qquad (n = 2, 3, \cdots),$$

$$H_1 = \int \varphi_1\bar{\eta}, \qquad H_n = \begin{vmatrix} \int \varphi_1\bar{\varphi}_1 & \cdots & \int \varphi_1\bar{\varphi}_{n-1} & \int \varphi_1\bar{\eta} \\ \int \varphi_2\bar{\varphi}_1 & \cdots & \int \varphi_2\bar{\varphi}_{n-1} & \int \varphi_2\bar{\eta} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int \varphi_n\bar{\varphi}_1 & \cdots & \int \varphi_n\bar{\varphi}_{n-1} & \int \varphi_n\bar{\eta} \end{vmatrix} \qquad (n = 2, 3, \cdots),$$

* It is easily seen that this theorem still holds if the condition of orthogonality is replaced by the less restrictive requirement that the real part of $\int \varphi_i\varphi_j$ vanish, i. e.,

$$\int \varphi_i\bar{\varphi}_j + \int \bar{\varphi}_i\varphi_j = 0 \qquad (i \neq j).$$

† Cf. equation (30) of (I). The notation used above corresponds with that of (I).

$$G_0 = 1, \qquad G_n = G(\varphi_1, \cdots, \varphi_n), \quad (n = 1, 2, 3, \cdots).$$

The ψ's are all mutually orthogonal functions of the class Ω; for as

$$\int \psi_n \varphi_i = \int \bar{\psi}_n \bar{\varphi}_i = 0 \qquad (i = 1, 2, \cdots, n-1),$$

we have

$$\int \psi_n \bar{\psi}_m = 0 \qquad (m \neq n).$$

Moreover we have the relations

$$\int |\psi_n|^2 = \int \psi_n \bar{\psi}_n = \int \psi_n \varphi_n G_{n-1} = G_n G_{n-1},$$

$$\int \psi_n \eta = \bar{H}_n, \qquad \int \bar{\psi}_n \bar{\eta} = H_n,$$

so that (27) may be put in the form

$$\xi_1^{(m)} = \bar{\eta} - \sum_{n=1}^{n=m} \frac{\int \bar{\psi}_n \bar{\eta}}{\int |\psi_n|^2} \psi_n. \tag{28}$$

From (5) we have

$$\text{norm } \xi_1^{(m)} = \int \eta \xi_1^{(m)} = \int |\eta|^2 - \sum_{n=1}^{n=m} \frac{|\int \psi_n \eta|^2}{\int |\psi_n|^2},$$

which shows that the series of positive or zero terms

$$\sum_{n=1}^{\infty} \frac{|\int \psi_n \eta|^2}{\int |\psi_n|^2}$$

is convergent for every function η of the class Ω since the sum of its first m terms is not greater than $\int |\eta|^2$. Now this series is precisely that formed by the norms of the terms of the series

$$\sum_{n=1}^{\infty} \frac{\int \bar{\psi}_n \bar{\eta}}{\int |\psi_n|^2} \psi_n, \tag{29}$$

whose terms are all mutually orthogonal; hence by Theorem 9 the series (29) converges in the mean to a function of the class Ω when η is a function of this class, as does likewise the series obtained from (28) by extending the summation to $n = \infty$. Thus we have shown the existence of a function $\xi_1 = \lim_{m=\infty} \xi_1^{(m)}$, and from (23) it is clear that ξ_1 is a solution of equations (26). Conversely, if ξ_1 is any solution of these equations we may obtain it by putting $\bar{\eta} = \xi_1$ in (28); for then all the terms after the first vanish. We therefore have

THEOREM 10. *If η is a function of the class Ω, the function $\xi_1^{(n)}$ given by formula* (4) *converges in the mean as n becomes infinite to a function ξ_1*

of the class Ω which is a solution of equations (26). *Conversely, every solution of* (26), *whether of the class Ω or not, can be obtained in this way by properly choosing η.*

The corollary to Theorem 9 shows that

$$\text{norm } \xi_1 = \text{norm } \eta - \sum_{n=1}^{\infty} \frac{\left|\int \psi_n \eta\right|^2}{\int |\psi_n|^2}. \tag{30}$$

Referring to (6) we see that we may also write

$$\text{norm } \xi_1 = \lim_{n=\infty} \frac{G(\eta, \varphi_1, \cdots, \varphi_n)}{G(\varphi_1, \cdots, \varphi_n)}. \tag{30'}$$

Definition. *A system of functions of the class Ω is termed complete in the interval (a, b) if there exists no other function of this class, not essentially zero, which is orthogonal to all the functions of the system in this interval.*

If the system $\{\varphi_n\}$ is complete the equations (26) have no solution not essentially zero; then norm $\xi_1 = 0$, and in view of (30) we have

Theorem 11. *A necessary and sufficient condition that the system of functions $\{\varphi_n\}$ of the class Ω be complete is that, for any function η of this class, we have*

$$\text{norm } \eta = \sum_{n=1}^{\infty} \frac{\left|\int \psi_n \eta\right|^2}{\int |\psi_n|^2}, \tag{31}$$

or

$$\lim_{n=\infty} \frac{G(\eta, \varphi_1, \cdots, \varphi_n)}{G(\varphi_1, \cdots, \varphi_n)} = 0. \tag{32}$$

In terms of the normalized orthogonal system $\{\Phi_n\}$ formed from $\{\varphi_n\}$ as indicated by (17), the condition (31) takes the form

$$\text{norm } \eta = \sum_{n=1}^{\infty} \left|\int \Phi_n \bar{\eta}\right|^2 \tag{33}$$

since

$$\frac{\left|\int \psi_n \eta\right|^2}{\int |\psi_n|^2} = \frac{|H_n|^2}{G_{n-1} G_n} = \left|\int \Phi_n \bar{\eta}\right|^2.$$

If the system $\{\varphi_n\}$ is incomplete we see from (30) that

$$\text{norm } \eta \geqq \sum_{n=1}^{\infty} \left|\int \Phi_n \bar{\eta}\right|^2 \tag{34}$$

so that the series on the right is convergent whenever η is of the class Ω.

In the important case in which $\{\varphi_n\}$ is a normalized orthogonal system we may put $\Phi_n = \varphi_n$ in (33) and (34), which then yield the well-known *Vollständigkeitsbedingung* and *Bessel's Inequality* respectively.

7. We proceed to apply the foregoing results to the problem of expanding an arbitrary function in terms of a preassigned system of functions by the method of least squares.*

THEOREM 12.† *Let θ be any function of the class Ω and $\{\varphi_k\}$ a complete system of functions of this class none of which are essentially linearly dependent. Then if the constants a_{nk} in the finite series*

$$\sigma_n = \sum_{k=1}^{k=n} a_{nk}\varphi_k \qquad (n = 1, 2, \cdots),$$

are chosen so that norm $(\theta - \sigma_n)$ has the smallest value,

$$\lim_{n=\infty} \sigma_n = \theta.$$

Writing

$$a_{nk} = a_{nk}' + ia_{nk}'', \qquad \bar{a}_{nk} = a_{nk}' - ia_{nk}'' \quad (k = 1, 2, \cdots, n),$$

and employing the classical procedure to find the values of the $2n$ constants, a_{nk}', a_{nk}'' which render

$$I_n = \text{norm } (\theta - \sigma_n) = \int (\theta - \sigma_n)(\bar{\theta} - \bar{\sigma}_n)$$

a minimum, we find that

$$(35) \qquad \begin{cases} \dfrac{\partial I_n}{\partial a_{nk}'} = -\displaystyle\int [(\theta - \sigma_n)\bar{\varphi}_k + (\bar{\theta} - \bar{\sigma}_n)\varphi_k] = 0, \\ \dfrac{\partial I_n}{\partial a_{nk}''} = i\displaystyle\int [(\theta - \sigma_n)\bar{\varphi}_k - (\bar{\theta} - \bar{\sigma}_n)\varphi_k] = 0; \end{cases} \qquad (k = 1, 2, \cdots, n)$$

$$(36) \qquad \frac{\partial^2 I_n}{\partial a_{nk}'^2} = \frac{\partial^2 I_n}{\partial a_{nk}''^2} = 2\int \varphi_k\bar{\varphi}_k > 0, \qquad \frac{\partial^2 I_n}{\partial a_{nk}'\partial a_{nk}''} = 0,$$

noticing that φ_k is not essentially zero as none of the φ's are essentially linearly dependent. Now equations (35) are equivalent to

$$\begin{aligned} \int (\theta - \sigma_n)\bar{\varphi}_k &= 0, \\ \int (\bar{\theta} - \bar{\sigma}_n)\varphi_k &= 0, \end{aligned} \qquad (k = 1, 2, \cdots, n)$$

either set of which follows from the other. Thus $\bar{\theta} - \sigma_n$, which is a linear combination of $\bar{\theta}$ and the $\bar{\varphi}$'s, is a solution of the homogeneous equations $\int \varphi_k\xi = 0$ $(k = 1, 2, \cdots, n)$, and is therefore uniquely determined by formula (4) upon putting $\eta = \theta$. In view of (36) this expression for $\bar{\theta} - \bar{\sigma}_n$ gives I_n its minimum value, which by (6) is

* See Gram, Über die Entwickelung reeler Funktionen in Reihen mittelst der Methode der kleinsten Quadrate, Crelle, 94 (1883), pp. 41–73.

Byerly, "Approximate Representation," Annals of Mathematics, Ser. 2, 12 (1911), pp. 128–148.

† Cf. Gram, l. c., p. 59.

$$\text{norm } (\theta - \sigma_n) = \frac{G(\theta, \varphi_1, \cdots, \varphi_n)}{G(\varphi_1, \cdots, \varphi_n)}.$$

Since the system $\{\varphi_k\}$ is complete we have from Theorem 11 that

$$\lim_{n=\infty} \text{norm } (\theta - \sigma_n) = 0,$$

or, in our notation, $\lim_{n=\infty} \sigma_n = \theta$, as we wished to prove.

8. Lastly we consider the infinite system of non-homogeneous equations

$$\text{(37)} \qquad \int \varphi_1 \xi = b_1, \qquad \int \varphi_2 \xi = b_2, \qquad \cdots,$$

where we again assume that $\varphi_1, \varphi_2, \cdots$ are all of the class Ω and none of them essentially linearly dependent. The principal solution of the first n of these equations, which we will denote by $\xi_0^{(n)}$, is given by formula (9), and an argument similar to that used in (I), p. 182, shows that we may write $\xi_0^{(m)}$ as the sum of m mutually orthogonal functions of the class Ω,

$$\xi_0^{(m)} = \sum_{n=1}^{n=m} \frac{B_n}{G_{n-1}G_n} \psi_n,$$

where ψ_n is defined as in § 6 and

$$B_1 = b_1, \qquad B_n = \begin{vmatrix} \int \varphi_1 \bar{\varphi}_1 & \cdots & \int \varphi_1 \bar{\varphi}_{n-1} & b_1 \\ \int \varphi_2 \bar{\varphi}_1 & \cdots & \int \varphi_2 \bar{\varphi}_{n-1} & b_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int \varphi_n \bar{\varphi}_1 & \cdots & \int \varphi_n \bar{\varphi}_{n-1} & b_n \end{vmatrix} \qquad (n = 2, 3, \cdots).$$

Theorem 9 shows that $\xi_0^{(m)}$ will converge in the mean to a function of the class Ω when and only when the norms of the terms of the series

$$\sum_{n=1}^{\infty} \frac{B_n}{G_{n-1}G_n} \psi_n$$

form a convergent series. Thus when

$$\text{(38)} \qquad \sum_{n=1}^{\infty} \frac{|B_n|^2}{G_{n-1}G_n}$$

converges there exists a function $\xi_0 = \lim_{n=\infty} \xi_0^{(n)}$ of the class Ω, and from (23) we see that ξ_0 is a solution of (37). Now if the equations (37) have *any* solution, ξ, of the class Ω, then, $\xi_0^{(m)}$ being the solution of least norm of the first m of these equations,

$$\text{norm } \xi_0^{(m)} \leqq \text{norm } \xi;$$

and as norm $\xi_0^{(m)}$ is precisely the sum of the first m terms of series (38), whose terms are never negative, this series converges. Thus we have proved

THEOREM 13. *A necessary and sufficient condition that equations* (37) *have a solution of the class* Ω *is that series* (38) *converge. When this is the case* $\xi_0^{(n)}$, *given by formula* (9), *converges in the mean to a function of the class* Ω *as n becomes infinite, and this function* ξ_0 *is a solution of equations* (37).

ξ_0 is termed the *principal solution* of (37). We may form the general solution ξ by adding to ξ_0 the general solution ξ_1 of equations (26): $\xi=\xi_0+\xi_1$. If the system $\{\varphi_n\}$ is complete ξ_1 is essentially zero; then if the series (38) converge the equations (37) have essentially but one solution, the principal solution.

If ξ_1 is a function of the class Ω, ξ is likewise and its norm is given by

$$\text{norm } \xi = \text{norm } \xi_0 + \text{norm } \xi_1;$$

for from (8) we have $\int \xi_0^{(n)}\bar{\xi}_1 = 0$, so that upon applying (23), $\int \xi_0\bar{\xi}_1 = 0$. Thus we have

THEOREM 14. *Among the solutions of* (37) *no others have so small a norm as the principal solution except those essentially equal to it.*

The norm of ξ_0 is given by the series (38), as is clear from the Corollary to Theorem 9; or we may write

$$\text{norm } \xi_0 = \lim_{n=\infty} - \frac{\begin{vmatrix} \int \varphi_1\bar{\varphi}_1 & \cdots & \int \varphi_1\bar{\varphi}_n & b_1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \int \varphi_n\bar{\varphi}_1 & \cdots & \int \varphi_n\bar{\varphi}_n & b_n \\ \bar{b}_1 & \cdots & \bar{b}_n & 0 \end{vmatrix}}{G(\varphi_1, \cdots, \varphi_n)}. \tag{39}$$

An important case arises when, in (37), the functions $\{\varphi_n\}$ form a normalized orthogonal system, when, in particular, none of them will be essentially linearly dependent. Then $B_n = b_n$ and series (38) becomes $\Sigma|b_n|^2$; Theorem 13 for this case is the *Theorem of Riesz and Fischer.**

DEFINITION. *Two systems of functions,* $\{\varphi_n\}$ *and* $\{\psi_n\}$, *of the class* Ω *are said to form a biorthogonal system* $\{\varphi_n, \psi_n\}$ *in the interval* $a \leqq x \leqq b$ *if*

$$\int_a^b \varphi_i\bar{\psi}_j dx = \int_a^b \bar{\varphi}_i\psi_j dx = \begin{cases} 1 & (i = j), \\ 0 & (i \neq j). \end{cases}$$

Each of the systems $\{\varphi_n\}$, $\{\psi_n\}$ *is called the adjoint of the other.*

* F. Riesz, Comptes Rendus, April, 1907, p. 734; E. Fischer, Comptes Rendus, May, 1907, p. 1022.

If $\{\varphi_n, \psi_n\}$ is a biorthogonal system of functions, none of the functions of $\{\varphi_n\}$ or of $\{\psi_n\}$ can be essentially linearly dependent; if, for example,

$$c_1\varphi_1 + c_2\varphi_2 + \cdots + c_k\varphi_k = 0$$

(essentially), we have, upon multiplying this relation by $\bar{\psi}_i$ and integrating from a to b, $c_i = 0$ $(i = 1, 2, \cdots, k)$.

Now let $\{\varphi_n\}$ be a system of functions of the class Ω none of which are essentially linearly dependent. We inquire under what condition $\{\varphi_n\}$ will have an adjoint system; or when the system of equations

$$\int \varphi_i \xi = \begin{cases} 1 & (i = n), \\ 0 & (i \neq n), \end{cases} \tag{40}$$

has a solution of the class Ω for $n = 1, 2, 3, \cdots$. A necessary and sufficient condition that (40) have such a solution is that the series (38) converge for

$$b_i = \begin{cases} 1 & (i = n), \\ 0 & (i \neq n); \end{cases}$$

this series, in the notation of § 4, reduces to

$$\frac{G_{n-1}}{G_n} + \frac{|G_{n+1}{}^{(n)}|^2}{G_n G_{n+1}} + \frac{|G_{n+2}{}^{(n)}|^2}{G_{n+1}G_{n+2}} + \cdots$$

or with regard to (18),

$$a_{nn}{}^2 + |a_{n+1,\,n}|^2 + |a_{n+2,\,n}|^2 + \cdots. \tag{41}$$

Thus we have proved a theorem due to A. J. Pell:* *A necessary and sufficient condition that a system of function* $\{\varphi_n\}$ *of the class* Ω, *none of which are essentially linearly dependent, have an adjoint system is that the series* (41) *converge for* $n = 1, 2, \cdots$.

When the conditions of this theorem are fulfilled and the system $\{\varphi_n\}$ is complete the adjoint system will be essentially unique.

9. The expressions (30)′ and (39) for the norms of ξ_1 and ξ_0 suggest the problem of determining under what conditions the determinants in the numerator and denominator converge as n becomes infinite. As the quotient of these determinants converges we need merely inquire when the *infinite Gramian* $G(\varphi_1, \varphi_2, \cdots)$ converges. By means of Theorem 4 and the theorem concerning bordered Gramians stated in the footnote to equation (12) we may prove just as in (I), § 7, the following

* "Biorthogonal Systems of Functions," Transactions of the American Mathematical Society, 12 (1911), p. 141.

THEOREM 15. *A sufficient condition that the infinite Gramian of a system of essentially linearly independent functions* $\{\varphi_n\}$ *of the class* Ω *converge is that the infinite product* $\Pi_{n=1}^{\infty} \int |\varphi_n|^2$ *diverge to zero or converge.*

From the proof of this theorem the following corollaries follow immediately.

COROLLARY 1. *If the infinite Gramian of the normalized system of functions formed from* $\{\varphi_n\}$ *does not diverge to zero the condition that* $\Pi_{n=1}^{\infty} |\varphi_n|^2$ *diverge to zero or converge is also necessary for the convergence of* $G(\varphi_1, \varphi_2, \cdots)$.

COROLLARY 2. *If* $\Pi_{n=1}^{\infty} |\varphi_n|^2 = 0$, *then* $G(\varphi_1, \varphi_2, \cdots) = 0$.

UNIVERSITY OF CINCINNATI,
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THE METHOD OF MONODROMIE WITH APPLICATIONS TO THREE PARAMETER QUARTIC EQUATIONS.

By R. P. Baker.

§ 1. The Technique of Monodromie.

The fundamental theorem on this subject is given by Jordan in his Traité des Substitutions, p. 277. It is there shown that, if $F(x, k) = 0$ is an algebraic equation, with coefficients in a domain including k, and has the algebraic group G, then a group of substitutions H can be found such that every rational function of the roots and k which is invariant under G is invariant under H; that H is an invariant subgroup under G and that an algebraic number can always be found whose adjunction reduces G to H. Moreover the method of determining H and this algebraic number are formally set out.

For the determination of H the parameter is allowed to vary in such a way that its path on the Neumann sphere for the complex variable k is a closed curve which does not pass through the branch points. Such a variation of k may cause the roots $(x_1, x_2, \cdots x_n)$ to be permuted according to a certain substitution which is an element of a group H generated by the totality of such substitutions. H is called the monodromie group.

The method of determining the algebraic number t whose adjunction reduces G to H is as follows: If φ is a function of the roots which is invariant under H and changed by every other substitution, it is a rational function of k, say $f(k)$. Replace all the coefficients of $f(k)$ by indeterminates and substitute the expression in the irreducible equation for φ, and then express the fact that the result is zero for every k. A certain number of algebraic equations are obtained with coefficients in R, which must be satisfied by the coefficients of $f(k)$. These coefficients are the roots of numerical equations in R, and their adjunction reduces G to H. This adjunction may be replaced by the adjunction of a single algebraic number, or by adjunction of all the roots of a single equation. From the latter method it follows that H is invariant under G.

The effective determination of H calls for a method of determining a finite number of elements, which generate the whole group. This is the object of this paper, the illustrations being taken from quartic equations. Incidentally the determination of t is illustrated.

§ 2. The Effective Determination of the Monodromie Group.

In the case of one parameter, k, the discriminant of the equation is a polynomial in k. At only a finite number of points are the roots equal. These points must be deleted from the domain of k, for otherwise the roots lose their identity and no question of a substitution can arise. Consider a path for k made up of a line from k_0, a non-discriminantal point, to a point arbitrarily near k_1, a point at which two roots are equal, a circle round k_1 and a return on the line to k_0. The line and the return can produce no substitution since the roots are all distinct and return to their original positions. The circle may or may not give a single transposition.

If in the neighborhood of k_1 the equation can be approximately expressed in the form

$$[(x - x_1)^2 - p(k - k_1)^n]G(x) = 0,$$

where n (> 1) is the index of the lowest power of $(k - k_1)$ entering, and $G(x)$ is finite as (x, k) approaches (x_1, k_1), then two cases arise:—

If n is an even integer $2m$, the first term reduces and, as $(k - k_1)$ expressed in the form ρ Exp $i(\varphi + \alpha)$ varies as φ increases from 0 to 2π, the two points representing the values of $(x - x_1)$, which are as nearly as we please $\pm \sqrt{\rho p} \cdot \text{Exp}\, mi(\varphi + \alpha)$, each make m circuits round the origin and are restored to their original position. In this case no substitution occurs.

If however n is an odd integer $2m + 1$, the values of $(x - x_1)$ are as nearly as we please $\pm \sqrt{\rho p} \cdot \text{Exp}\, (m + \frac{1}{2})i(\varphi + \alpha)$ and each makes m circuits and a half circuit, and the two roots suffer a simple interchange. The remaining roots, satisfying as nearly as we please the equation $G(x) = 0$, are finitely separated and remain each within a region that may be made as small as we please, and so cannot be interchanged.

In the general case a discriminantal value of k may cause several sets of roots to become equal, the irreducible binomials giving the cycles of the substitutions.

The most convenient method of determining the proper approximation is the Newton parallelogram. It may happen that a determination of the branch to a higher order is necessary, the details being worked out as in Chrystal's Algebra, vol. II, p. 383.

§ 3. A General Theorem concerning Monodromie.

Single transpositions occur at ordinary points on the discriminantal locus; all other substitutions including the identity occur at singular points.

If the roots are $x_1, x_2, x_3, \cdots x_n$ and Δ is the discriminant, the elementary symmetric functions being a_1, a_2, a_3, a_n, and if $\zeta(1, 2, \cdots n)$ denote

the product of the differences of the roots taken in ascending order of subscripts, then giving a_1 an increment α and in consequence the roots increments $\xi_1, \xi_2, \cdots \xi_n$, we have, to the first order in small quantities,

$$\begin{aligned}
\alpha &= \xi_1 + \xi_2 + \xi_3 + \cdots + \xi_n \\
0 &= \xi_1 \cdot \Sigma x_2 + \xi_2 \cdot \Sigma x_1 + \cdots \\
0 &= \xi_1 \cdot \Sigma x_2 x_3 + \xi_2 \cdot \Sigma x_1 x_3 + \cdots \\
&\cdots\cdots\cdots\cdots \\
0 &= \xi_1 \cdot x_2 x_3 \cdots x_n + \xi_2 \cdot x_1 x_3 \cdots x_n + \cdots,
\end{aligned}$$

where the subscripts of the x's omit the subscript of the ξ which multiplies their summations.

By solving this set of linear equations and proceeding to the limit, there results

$$\frac{\partial x_1}{\partial a_1} = \frac{x_1^{n-1}\zeta(2, 3, 4, \cdots n)}{\zeta(1, 2, 3, \cdots n)},$$

where the numerator ζ is defined as the same function of the set of roots omitting x_1 as the denominator is of the whole set.

In general by the same method we have

$$\frac{\partial x_j}{\partial a_k} = \frac{\zeta(1, 2, \cdots j-1, j+1, \cdots n)}{\zeta(1, 2, \cdots n)} \cdot x_j^{n-k}.$$

From this, since $\Delta = \zeta^2$,

$$\frac{\partial \Delta}{\partial a_k} = \Sigma \frac{\partial \Delta}{\partial x_j} \frac{\partial x_j}{\partial a_k} = 2\Sigma \frac{\partial \zeta}{\partial x_j} x_j^{n-k}\zeta(1, 2, \cdots j-1, j+1, \cdots n).$$

If more than two roots are equal or if there are two or more pairs of equal roots, every one of the partial derivatives in the last expression vanishes, and therefore every partial derivative of Δ with respect to the coefficients vanishes. Hence the point is singular on Δ. This establishes a necessary condition for substitutions other than single transpositions.

To arrive at sufficient conditions we suppose that the factor Δ_1 of the discriminant vanishes at the point $a_1', a_2', \cdots a_n'$ and the roots in its neighborhood form certain sets of equal roots. Since the values of the roots and the coefficients are analytic functions of Δ_1 with convergent series for small enough values of Δ_1, the equation can be expressed as nearly as we please in the form

$$F(x) = [(x - x_1)^{\alpha_1} - p_1\Delta_1^{\delta_1}][(x - x_2)^{\alpha_2} - p_2\Delta_1^{\delta_2}][\cdots] \cdots G(x) = 0,$$

where $G(x) = 0$ has no equal roots.

Taking first the case of the α's prime to the δ's, the vanishing terms of the discriminant in the field of the roots are contributed by differences of roots in the same cycle. If α is odd the discriminant of the binomial is even in Δ_1^δ and hence even in Δ_1. If α is even δ must be odd and the discriminant of the binomial is odd. Now the odd α's contribute odd cycles, equivalent to an even number of transpositions, and the even α's even cycles, hence if the whole substitution is odd the exponent of Δ_1 in the discriminant is odd; if even, even.

Taking next the case of an α and a δ having a common factor, we have several cycles with the same root value. Let these be

$$[(x - x_1)^{\alpha_1} - p_1\Delta_1^{\delta_1}][(x - x_1)^{\alpha_2} - p_2\Delta_1^{\delta_2}]\,[\cdots]\,\cdots.$$

The equalities among the roots of the same cycle obey the odd and even law as above and, if

$$\frac{\alpha_1}{\delta_1} \geqq \frac{\alpha_2}{\delta_2} \geqq \frac{\alpha_3}{\delta_3} \geqq \cdots,$$

the differences of the roots in the first cycle taken with those of the following cycles are in number $\alpha_1(\alpha_2 + \alpha_3 + \alpha_4 + \cdots)$ and of order δ_1/α_1, and the contribution from their squares to the order of Δ_1 in the discriminant is an even integer. The same holds of the pairs from the second cycle and those that follow it, and so on. The odd and even rule will be kept in this case and also if any number of root values have compound cycles. The rule then holds in general.

We notice that a pair of linear factors contribute at least Δ_1^2. This is the case corresponding to a nodal locus where no cycle exists.

Also if the δ's are known explicitly for all the cycles the exponent can be calculated; and that each set of cycles has a minimum exponent corresponding to the case where all the δ's are equal to unity. In the case of several parameters fixed values, not discriminantal and not causing reduction, may be given to all but one, and the cycles discovered by its variation are elements of the monodromie group. But if a discriminantal value or reducing value is assigned as fixed, in general cycles will be lost.

§ 4. The General Quartic.

By a rational transformation this can be brought to the form

$$x^4 + bx^2 - cx + d = 0. \tag{1}$$

The discriminant is

$$\Delta = 256d^3 + 16b^4d + 144bc^2d - 4b^3c^2 - 128b^2d^2 - 27c^4. \tag{2}$$

The discriminantal surface has the singular lines

$$c = 0, \qquad b^2 - 4d = 0,$$

and

$$b^2 + 12d = 0, \qquad 8b^3 + 27c^2 = 0.$$

The first is a locus of two pairs of equal roots, the second of three equal roots. The origin is a singular point at which four roots are equal. (Weber's Algebra, I, 249.) There are no other singularities.

To show that the corresponding cycles occur, we choose for the first line the point $c = 0$, $b = 2$, $d = 1$. The roots are $+ i$, $- i$, each repeated. Writing $x = \xi + i$, $c = \gamma$, $b = 2 + \beta$, $d = 1 + \delta$, the equation becomes

$$\xi^4 + 4i\xi^3 + \xi^2(\beta - 4) + \xi(2i\beta - \gamma) + \delta - \beta = 0.$$

For $\beta = 0$, $\gamma = 0$, which are not in themselves discriminantal, the Newton parallelogram gives as proper approximation $-4\xi^2 + \delta = 0$ and a simple transposition of the pair of equal roots follows from the monodromie of δ round its origin. The same thing occurs for the other pair of equal roots.

It may be noticed that with $\beta = 0$, $\delta = 0$ reduction occurs and variations of γ can give no cycle.

For the locus of three equal roots we write $x = \xi + \sqrt{2}$, $b = - 12$, $c = - 16\sqrt{2}$ and $d = - 12 + \delta$.

We have

$$\xi^4 + 4\sqrt{2}\xi^3 - \delta = 0,$$

and a 3-cycle. At the origin, writing $x = \xi$, $d = \delta$, $b = c = 0$, $\xi^4 + \delta = 0$ gives a 4-cycle.

Since no group but the symmetric group contains substitutions of these three types, the monodromie group is the symmetric group. In this case it follows at once that the algebraic group is the symmetric group.

§ 5. Quartics with the Alternating Group.

To find a general form for these, we express the fact that the discriminant is the square of a rational quantity and, by a chain of birational substitutions, reduce this equation to a form in which the coefficients can be determined as rational functions of prescribed form of three parameters.

The form thus arrived at will have generality of this kind, viz: It is a three parameter form, every special case of which has a square discriminant, and so a group G_{12}, or a subgroup, provided the discriminant is not zero. If as is usual irreducibility is postulated, it is general in the same sense as the general equation is in reference to its group. For every form with a non-vanishing square discriminant can be identified with this one. It turns out that the form possesses a property which the general equation does not, namely that a parameter can be removed by a rational transformation.

The discriminant can be written

$$16d(4d - b^2)^2 + 36bc^2(4d - b^2) + 32b^3c^2 - 27c^4 = t^2.$$

Putting $4d - b^2 = \rho$ and eliminating d, a birational operation, we have

$$t^2 = 4(b^2 + \rho)\rho^2 + (36b\rho + 32b^3 - 27c^2)c^2.$$

Write $t = 4nr$, $c = 2mr$, $\rho = 2r$ and eliminate c, t, ρ. The result, arranged as a quadratic in r, is then

$$27m^4r^2 - 2r(9bm^2 + 1) + n^2 - b^2 - 8m^2b^3 = 0.$$

If r is rational the discriminant must be a square.

Writing $3bm^2 = \eta$, $3nm^2 = \sigma$, this gives

$$8\eta^3 + 12\eta^2 + 6\eta + 1 - 3\sigma^2 = q^2,$$

or, if $2\eta + 1 = \theta$,

$$\theta^3 - 3\sigma^2 = q^2.$$

Apply now the birational transformation,

$$\sigma = s\theta, \qquad q = p\theta,$$

and obtain

$$\theta = 3s^2 + p^2.$$

Retracing the chain of transformations,

$$\begin{aligned} b &= \frac{(3s^2 + p^2 - 1)}{6m^2}, \\ c &= \frac{(3 + 2p)(3s^2 + p^2) - 1}{27m^3}, \\ d &= \frac{4[(3 + 2p)(3s^2 + p^2) - 1] + 3[3s^2 + p^2 - 1]^2}{432m^4}, \\ t &= \frac{2s(3s^2 + p^2)[(3 + 2p)(3s^2 + p^2) - 1]}{81m^6}. \end{aligned} \tag{3}$$

It is obvious that m can be removed by a rational transformation. In the course of the work the factors r^2 and θ^2 are rejected. Their vanishing leads to reduction.

To determine the monodromie group for this form, we remark that the parameter m is ineffective. Removing it by writing $x = \xi/6m$, and, in order to discuss the factor s of the discriminant, placing $p = 1$, we have in the neighborhood of the equal roots, if $\xi = 2 + y$ and $s = \sigma$, where y and σ are small,

$$81\sigma^4 + \sigma^2(18y^2 - 4\delta y + 12) + (y^4 + \delta y^3 + 24y^2) = 0$$

or, with the proper approximation

$$2y^2 + \sigma^2 = 0.$$

The locus $s = 0$ is then a nodal locus and gives no cycles. To test the locus $3s^2 + p^2 = 0$, we write $3s^2 = -1$, $p = 1 + \pi$, where π is small. The equal roots are at $\xi = 1$ and, writing $\xi = 1 + y$ where y is small, the proper approximation is

$$y^3 + 4\pi = 0.$$

Here three roots enter a cycle.

For the locus

$$(2p + 3)(3s^2 + p^2) - 1 = 0$$

take

$$p = 0, \qquad s = \tfrac{1}{3} + \lambda, \qquad \xi = \sqrt{2} + y.$$

The approximation is

$$y^2 + 3(3 - 2\sqrt{2})\lambda = 0.$$

This gives a cycle of two roots and similarly for the values $\xi = -\sqrt{2}$. The element is a double transposition. The only groups containing cycles of three and double transpositions are the symmetric group and the alternating group.

If in the investigation the exact form of the discriminant is known, the application of the method is trivial; if, however, as may be conceived, only the algebraic factors of the discriminant are known, the complete determination of the monodromie group requires the discussion of the intersection of the loci and their singularities. In this case a moderate amount of labor, which is left to the reader, shows that no further type of cycles arises and the fact that all the triple cycles and double transpositions occur in the alternating group obviates any need of identifying the roots.

With the given form the algebraic group is the alternating group. If we write $s = \sqrt{2}\sigma$ the coefficients will not contain the square root but the algebraic group is now the general group, reducing on adjunction of $\sqrt{2}$ to the alternating group in accordance with Jordan's theorem.

§ 6. Quartic Equations with the Group G_8.

The equation being as before (1), and the function $x_1x_2 + x_3x_4$ being rational, the equation satisfied by this and its conjugates, $x_1x_3+x_3x_4=t_2$, $x_1x_4 + x_3x_2 = t_3$, is

$$t^3 - bt^2 - 4dt + 4bd - c^2 = 0.$$

This is linear in b and the general equation with this group can be written

$$x^4 + \left(t - \frac{c^2}{\rho}\right)x^2 - cx + \left(\frac{t^2 - \rho}{4}\right) = 0, \tag{4}$$

since no restrictions on rationality are introduced by the transformation $t^2 - 4d = \rho$ which is birational in (d, ρ).

The discriminant is equal to the discriminant of the resolvent for

$$t_1 - t_2 = (x_1 - x_4)(x_2 - x_3),$$
$$t_2 - t_3 = (x_1 - x_2)(x_3 - x_4),$$
$$t_3 - t_1 = (x_1 - x_3)(x_4 - x_2).$$

But $(t_2 - t_3)^2$ can be expressed rationally, hence the discriminant reduces in the coefficients.

Corresponding to this factor we have Δ_1/ρ^2, where

$$\Delta_1 = 4t^2\rho^2 - 4c^2t\rho + c^4 - 4\rho^3. \tag{5}$$

The remaining factor is $\Delta_2^2 \div \rho^2$, where

$$\Delta_2 = \rho^2 + 2c^2t. \tag{6}$$

The complete discriminant is

$$D = \Delta_1 \cdot \Delta_2^2 \div \rho^4. \tag{7}$$

There are no singularities on either Δ_1 or Δ_2 at which $\rho \neq 0$. The discussion of monodromie may be limited to points on Δ_1, Δ_2, $\rho = 0$ and their intersections, and also $\rho = \infty$.

The locus $\Delta_2 = 0$ is a nodal locus and contributes no cycles or substitutions. The approximation at the point ($\rho = 2$, $t = -2$, $c = 1 + \gamma$, $x = \xi - 1$) is

$$7\xi^2 + 2\gamma\xi - \gamma^2 = 0.$$

This character is kept over the whole locus except possibly at the intersections with the other discriminantal loci.

The locus $\Delta_1 = 0$ produces a single transposition; the point ($\rho = 1$, $t = 3/2$, $c = 1 + \gamma$, $x = \xi + 1/2$ has for proper approximation

$$5\xi^2 - 2\gamma = 0.$$

The locus $\rho = 0$ gives a pair of transpositions, one pair of roots being zero and the other infinite. An exception arises if $c = 0$, for then $\Delta_2 = 0$ and no cycle is formed for a general approach.

The locus, $\rho = \infty$, gives a cycle of four roots with infinite values unless $c = 0$ when in general no cycle exists.

We may name the transposition on Δ_1 (12), and, by the continuity of the roots as we move along Δ_1 to $\rho = \infty$, $c \neq 0$, this pair of roots must take

for a real path the real (opposite) positions in the cycle of four. This cycle is then (1324) or (1423).

With these elements the group G_8 is generated and it remains to be shown that no further elements can occur to raise this to G_{24}.

As the problem stands, the labor of discussing all the intersections, including the infinite values, is prohibitive but can be avoided.

By means of a birational transformation of the parameters and a rational transformation of the variable, a parameter can be eliminated. We write

$$2t = \epsilon\pi(1 + \pi\lambda^2), \qquad c = \epsilon\lambda\pi^2, \qquad \rho = \epsilon\pi^2,$$

whose inverses are

$$\lambda = \frac{c}{\rho}, \qquad \pi = \frac{\rho^2}{2}(t\rho - c^2), \qquad \epsilon = \frac{4(t\rho - c^2)^2}{\rho^3}.$$

This gives for the coefficients

$$2b = \epsilon\pi(1 - \pi\lambda^2), \qquad 16d = \epsilon\pi^2[\epsilon(1 + \pi\lambda^2)^2 - 4].$$

If we now write $x = \xi/\lambda$ and $\pi\lambda^2 = \alpha$ the equation becomes

$$16\xi^4 + 8\epsilon\alpha(1 - \alpha)\xi^2 - 16\epsilon\alpha^2\xi + \epsilon\alpha^2[\epsilon(1 + \alpha)^2 - 4] = 0. \tag{8}$$

and the discriminant is

$$D = \epsilon^3\alpha^6(\epsilon\alpha^2 + \epsilon\alpha + 1)^2(\epsilon - 4).$$

It is now easy to show that, for $\alpha = -1$, G_8 is generated by the substitutions at ϵ cycles. The monodromie group is then either G_8 or G_{24}. For $\epsilon = 1$, the monodromie group is G_2', the only cycle being a double transposition at $\alpha = 0$. By Jordan's theorem the algebraic group can now be only the cyclic group in the G_8 or reduction has occurred. As a matter of fact the equation reduces and the question is at once raised as to the adjunction of a square root. Writing $\epsilon = \varphi^2$, the equation reduces to $\left[\xi^2 + \varphi\alpha\xi + \frac{\varphi^2\alpha(1 + \alpha + 2\varphi)}{4}\right]$ and the conjugate factor.

The adjunction of a square root cannot reduce the group from G_{24}, except to G_{12} which is barred since odd substitutions occur. Hence the group is G_8 and the monodromie group is the same.

Although more facts were at hand, the result could be obtained from a knowledge of the algebraic factors of the discriminant apart from their exponents and a monodromie in which identification of the roots was not needed.

§ 7. **Quartic Equations with the Group G_4: (1), (12) (34), (13) (24), (14) (23).**

As functions belonging to the group we may take any one of

$f_1 = (x_1 - x_2)(x_3 - x_4)$, $f_2 = (x_1 - x_3)(x_4 - x_2)$, $f_3 = (x_1 - x_4)(x_2 - x_3)$,

or their negatives.

If

$$t_1 = x_1x_2 + x_3x_4, \quad t_2 = x_1x_3 + x_2x_4, \quad t_3 = x_1x_4 + x_2x_3,$$

the t's are functions in G_8.

Since $f_1 = t_2 - t_3$ and the t's are roots of a cubic resolvent, then, if f_1 is rational, all the roots of this cubic are rational; for $t_2 - t_3$ is a six-valued function. It follows that all the f's are rational.

If the resolvent for t has one root, t_3, rational and $t_1 + t_2 = R$, the residual quadratic for t_1, t_2 is

$$T^2 - RT - \left(\frac{c^2}{R} + t_3^2 + Rt_3\right) = 0.$$

That this may have rational roots, the discriminant must be a square. This condition requires

$$R = 4c^2 \div (\kappa^2 - \lambda^2) \qquad \text{and} \qquad t_3 = \frac{(\lambda - R)}{2},$$

the discriminant being κ^2.

We may now take

$$t_1 = \frac{2c^2}{\kappa^2 - \lambda^2} + \frac{\kappa}{2}, \qquad t_2 = t_1 - \kappa, \qquad t_3 = \frac{\lambda}{2} - \frac{2c^2}{\kappa^2 - \lambda^2},$$

$$f_1 = \frac{4c^2}{\kappa^2 - \lambda^2} - \frac{\kappa + \lambda}{2}, \qquad f_2 = \frac{4c^2}{\kappa^2 - \lambda^2} + \frac{\kappa - \lambda}{2}, \qquad f_3 = \kappa.$$

The square root of the discriminant of the original equation is $-f_1f_2f_3$, hence the discriminant is

$$\Delta \equiv \frac{\kappa^2[64c^4 - 16\lambda(\kappa^2 - \lambda^2)c^2 + (\lambda^2 - \kappa^2)^3]^2}{16(\kappa^2 - \lambda^2)^4}. \tag{9}$$

By means of the given relations the equation becomes

$$x^4 + x^2\left[\frac{4c^2 + \lambda(\kappa^2 - \lambda^2)}{2(\kappa^2 - \lambda^2)}\right] - cx + \frac{[\kappa^2(\kappa^2 - \lambda^2)^2 - 8\lambda c^2(\kappa^2 - \lambda^2) + 16c^4]}{16(\kappa^2 - \lambda^2)^2} = 0. \tag{10}$$

The roots are

$$x_1 = \frac{c}{\sqrt{(\lambda^2 - \kappa^2)}} + \tfrac{1}{2}\sqrt{[-\lambda + \sqrt{(\lambda^2 - \kappa^2)}]}$$

and the conjugates.

The discriminantal locus, $\kappa = 0$, has a pair of roots equal to c/λ, and the approximation is of the form

$$x^4 + x^3 + x^2 + \kappa^2 x + \kappa^2 = 0,$$

where the coefficients, in general not zero, have been suppressed. No cycle is obtained.

If $\kappa = \lambda$, the discriminant is infinite. To examine this locus, write $\kappa - \lambda = \alpha$, and the equation is of the form

$$\alpha^2 x^4 + \alpha x^2 - \alpha^2 x + c^4 = 0.$$

If we write $x = 1/\xi$, the terms to be retained are

$$\alpha^2 + \alpha\xi^2 + \xi^4,$$

giving two separate cycles among the infinite vaues of x.

For the special values $c = 0$, $\lambda = -4$, the equation takes the form

$$x^4 - 2x^2 + \frac{\kappa^2}{16} = 0,$$

a real curve of the general type of a lemniscate, and it is easy to see that two different double transpositions occur and that the monodromie group contains G_4. The cycles occur here when $\kappa = \pm 4$, that is, $\kappa = \pm \lambda$. To show that the remaining factors of the discriminant contribute nothing further, we remark that

$$64c^4 - 16\lambda c^2(\kappa^2 - \lambda^2) + (\lambda^2 - \kappa^2)^3$$

reduces to

$$[8c^2 - (\kappa^2 - \lambda^2)(\lambda - \kappa)]\,[8c^2 - (\kappa^2 - \lambda^2)(\lambda + \kappa)],$$

that is, $4f_2 f_3(\kappa^2 - \lambda^2)^2$. Since the f's are all rational, there is no real distinction between the properties of $f_1 = 0$ and of $f_2 = 0$. Hence these factors are nodal also and contribute no cycles. From the point of view of monodromie only, this can be established by a special non-singular point which may be conveniently taken as

$$\lambda = 0, \qquad \kappa = -2, \qquad c = 1 + \gamma, \qquad x = \tfrac{1}{2} + \xi.$$

The approximate form is $x^2 + 2x^3 + 2x^2 + \gamma^2/2 \cdot x + \gamma^2/2$.

The detailed examination of the singularities of the discriminant may be shortened by the consideration that the intersection of two nodal loci cannot furnish cycles. This leaves the intersection of the effective loci $\kappa - \lambda = 0$, $\kappa + \lambda = 0$ as the only case to be examined for intersections and this with $c = 0$ as the only singularity on the nodal loci. It is easily shown that in no case does any other kind of substitution arise except double transposition.

If either of the quantities $\kappa - \lambda$, $\kappa + \lambda$ are restricted by being placed equal to a fixed rational quantity, the only remaining substitution is a

double transposition, and the monodromie group reduces to G_2, namely 1, (12)(34). If both are so restricted, that is, if κ, λ are constants, the variation of c produces no cycles and the monodromie group is the identity.

A two-parameter form for this case is obtainable from that for G_8 (8) by expressing the discriminant as a square.

The only non-square factor is $\epsilon(\epsilon - 4)$ and, by writing $\epsilon = 4/(1 - \lambda^2)$, we have the form

$$(1-\lambda^2)^2\xi^4 + 2\alpha(1-\alpha)(1-\lambda^2)\xi^2 - 4\alpha^2(1-\lambda^2)\xi + \alpha^2(\alpha^2+\lambda^2+2\alpha) = 0 \quad (11)$$

with the discriminant

$$4\lambda^2(1 - \lambda^2)^6\alpha^6[(2\alpha + 1)^2 - \lambda^2]^2.$$

The monodromie is managed by following the plan for the two-parameter form of the G_8 equation noting that the transformation from ϵ to λ destroys the single transpositions.

§ 8. The Reciprocal Quartic.

The equation being

$$x^4 - ax^3 + bx^2 - ax + 1 = 0, \quad (12)$$

the discriminant is

$$\Delta = -[a^2 - 4(b-2)]^2[4a^2 - (b+2)^2]. \quad (13)$$

If we assume irreducibility, the group may be determined thus: For a proper choice of root names, we have

$$x_1x_2 + x_3x_4 = 2.$$

The case of two conjugates being equal to 2 leads to the vanishing of the first factor of the discriminant, and, if one is 2 and the others are equal, the second factor vanishes. The group is then G_8 or a subgroup. The monodromie group contains a single transposition at $a = 1$, $b = 0$. The monodromie group and the algebraic group are then both G_8.

To show the inclusion of this case in the general form for G_8, we write the equation as

$$x^4 - 4ax^3 + 6bx^2 - 4ax + 1 = 0,$$

and remove the second term.

The result is

$$x^4 + 6(b - a^2)x^2 - 4a(2a^2 - 3b + 1)x + (1 + 6a^2b - 4a^2 - 3a^4) = 0,$$

which is included in the general form if we write

$$t = 2(1 - a^2), \quad \rho = -8a^2(3b - 2a^2 - 1), \quad c = -4a(3b - 2a^2 - 1).$$

To make a complete discussion by monodromie only, after establishing the single transposition, we consider a section of the locus by $b = 0$ and bring up for inspection complex points by writing

$$x = i\xi, \qquad a = -i\alpha.$$

The locus is then $\alpha = (1 + \xi^4)/\xi(1 - \xi^2)$ and the graph makes visible the two distinct double transpositions which are needed to generate the group G_8. The real graph shows that the two single transpositions may be given the same notation.

§ 9. Quartic Equations Solved by Quadratic Forms.

By a linear transformation these may be taken as

$$(x^2 + \beta)^2 + 2m(x^2 + \beta) + n = 0. \tag{14}$$

The discriminant is $64(m^2 - n)^2[(\beta + m)^2 - (m^2 - n)]$. If $\beta + m = -p$ and $m^2 - n = q$, the roots are

$$x_1 = \sqrt{p + \sqrt{q}}, \qquad x_2 = \sqrt{p - \sqrt{q}}, \qquad x_3 = -x_1, \qquad x_4 = -x_2.$$

The function $x_1x_3 + x_2x_4 = -2p$ is, in G_8, rational and distinct from its conjugates. The group is G_8 or a subgroup. Assuming irreducibility, it is G_8 if single transpositions occur.

For $\beta = 1$, $m = 1$, $n = 1 + \lambda$ and $x = \xi + i$, the approximation is

$$\xi^4 + 4i\xi^3 - 4\xi^2 + \lambda = 0$$

giving a 2-cycle and two distinct roots. The group is therefore G_8. The general form for G_8 equations includes this if

$$t = 2(\beta + m), \qquad c = 0, \qquad \rho = 4(m^2 - n).$$

To show by monodromie only that the group is at least G_8, we take the special case $\beta = 0$, when the equation reads

$$x^4 + 2mx^2 + n = 0.$$

If $m = 0$, we have a 4-cycle at the origin, if $m \neq 0$, a 2-cycle. By the continuity of the roots we see that these may be denoted by (1324) and (12) which generate G_8. To show that the group is not higher, we use the method of reduction of parameters. The special form just discussed is general by a birational transformation and change of notation.

§ 10. Quartic Equations with the Cyclic Group.

As a function belonging to $C_4[1 : (1234) : (13)(24) : (1432)]$, we take

$$(x_1 - x_3)(x_2 - x_4)[(x_1 - x_3)^2 - (x_2 - x_4)^2] = 4\lambda.$$

Assuming $a = x_1 + x_2 + x_3 + x_4 = 0$ and writing $x_1x_3 + x_2x_4 = t$, a function in G_8, we have

$$\lambda^2 = [t^2 - 4d][(t-b)^2 - 4t(t-b) + 16d], \tag{15}$$

where t is a rational solution of

$$t^3 - bt^2 - 4dt + 4bd - c^2 = 0. \tag{16}$$

The resolvent equation for λ, found by eliminating t, is

$$\begin{aligned}\lambda^6 - \lambda^4(16b^2d - 6bc^2 - 64d^2) + \lambda^2(9b^2c^4 - 384bc^2d - 32b^3c^2d + 108c^4d)\\ - c^4[256d^3 + 16b^4d + 144bc^2d - 4b^3c^2 - 128b^2d^2 - 27c^4] = 0.\end{aligned} \tag{17}$$

The last bracket contains the discriminant of the original equation.

To obtain the general equations of this group we write $t^2 - 4d = \rho$ and replace d in (15) and (16). This is a birational transformation as far as d and ρ are concerned.

From (16) we have $b = t - c^2/\rho$, and substitution in (15) gives

$$\lambda^2 = \frac{(2t\rho - c^2)^2}{\rho} - 4\rho^2.$$

Replace t by κ by the birational transformation $2t\rho - c^2 = \kappa\rho$, and we have $\lambda^2 + 4\rho^2 = \kappa^2\rho$. Using $\lambda = \sigma\rho$ to replace λ by σ, we have $\sigma^2\rho = \kappa^2 - 4\rho$. The factor ρ is removed in the operations, as $\rho = 0$ involves reduction. Solving for ρ and retracing the chain, we obtain b, d in terms of κ, σ, c, and the equation is

$$x^4 + x^2\left[\frac{\kappa^3 - c^2(\sigma^2+4)}{2\kappa^2}\right] - cx + \frac{[\kappa^3 + c^2(\sigma^2+4)]^2}{16\kappa^4} - \frac{\kappa^2}{4(\sigma^2+4)} = 0. \tag{18}$$

The discriminant for the integral form is

$$\Delta = \kappa^{18}\cdot(\sigma^2+4)^3\cdot\sigma^2[\kappa^6 + (\sigma^2+4)^2\kappa^3c^2 + (\sigma^2+4)^3c^4]^2,$$

and is wholly singular as it should be.

It is obvious that, if $\sigma^2 + 4$ is a square, the group reduces to G_4. In this case the equation reduces to two quadratics

$$x^2 + \frac{ctx}{\kappa} + \frac{t(\kappa^2 + c^2t^2) - 2\kappa^3}{4\kappa^2 t} = 0$$

and the conjugate, where

$$t^2 = \sigma^2 + 4.$$

As a check we notice that the cyclotomic equation is included for the values

$b = 5/8,\ c = -5/8,\ d = 205/256,\ t = 15/8,\ \kappa = 5/2,\ \sigma = 4,\ \lambda = 5/4.$ The group is C_4 for the conjugates are distinct, namely, if $\lambda = 0$ or if two values of t are equal, reduction occurs.

In the monodromie inquiry the nodal character of the squared bracket in the discriminant is conveniently established with the values $c = 10$, $\kappa = 1 + \epsilon$, $\sigma^2 = -81/20$ and $x = 1 + \xi$. For $\sigma = 2i$, four roots are infinite and a 4-cycle is formed; so also for the conjugate value of σ. Double transpositions occur at $\kappa = 0, \infty$ and at $\sigma = 0, \infty$.

The 4-cycle generates C_4 and the question whether the monodromie group is not higher remains.

As before the number of parameters may be reduced.

We write

$$c = \lambda\pi^2(\sigma^2 + 4), \qquad \kappa = \pi(\sigma^2 + 4),$$

and replace c, κ by π, λ which is a birational operation. Writing $x = \xi\sqrt{\lambda}$ and $\pi\lambda^2 = \alpha$ we have the form

$$\begin{aligned}16\xi^4 + 8\alpha(1-\alpha)(\sigma+4)\xi^2 - 16\alpha^2(\sigma^2+4)\xi \\ + \alpha^2(\sigma^2+4)[(1+\alpha)^2(\sigma^2+4) - 4] = 0\end{aligned} \tag{19}$$

with the discriminant

$$\Delta = (\sigma^2+4)^3\sigma^2\alpha^6[(\sigma^2+4)(\alpha^2+\alpha)+1]^2.$$

This differs from the two-parameter form for G_8 only by ϵ being replaced by $\sigma^2 + 4$. The single transposition of the G_8 form is now lost and the argument can be made as in the case of G_8.

We may notice as a convenient form of the resolvent equation for C_4 the condition that

$$16dt^2 - 3c^2t - 64d^2 - bc^2 \equiv \lambda^2$$

is a perfect square. This is derived by a partial elimination from (15) and (16).

§ 11.

As an explicit example of Jordan's method for determining the algebraic number whose adjunction reduces the group of the equation to the same group as the monodromie group, the case of a monodromie group G_4 under G_{12} may be taken.

To construct such an equation we notice that the factor $3s^2 + p^2$ of the discriminant of the G_{12} form (3) gives rise to 3-cycles. To obtain a G_4 monodromie group this must be obviated. It is sufficient for example to write $p = s$ The resulting form is

$$x^4 + 6(4s^2 - 1)x^2 - 8(8s^3 + 12s^2 - 1)x + 3(48s^4 + 32s^3 + 24s^2 - 1) = 0 \tag{20}$$

and the monodromie group is G_4.

The resolvent equation for G_4 when the discriminant Δ is a square is

$$f^3 - (12d + b^2)f - \sqrt{\Delta} = 0$$

in this case

$$b^2 + 12d = 2^6 \cdot 3^2(4s^2 + 2s + 1)s^2,$$

$$\sqrt{\Delta} = 2^9 \cdot 3^2(8s^3 + 12s^2 - 1)s^3.$$

By Jordan's theorem an algebraic field exists in which f is rational and in this special case all three f's are rational if one is (§ 7).

Writing $f = 8\varphi s$, we have for φ

$$\varphi^3 - 9(4s^2 + 2s + 1)\varphi - 9(8s^3 + 12s^2 - 1) = 0.$$

If now $\varphi = as + b$ and the equation is identically satisfied in s, we have

$$a^3 - 36a - 72 = 0, \qquad a^2b - 12b - 6a - 3b = 0,$$

$$ab^2 - 6b - 3a = 0, \qquad b^3 - 9b + 9 = 0.$$

These equations must by the general argument have consistent solutions.

Solving the equation for b we find

$$b_1 = 2\sqrt{3}\cos 50° = 2(\cos 80° + \cos 20°).$$

If ϵ is the primitive ninth root of unity, $\cos 40° + i \sin 40°$, and if $p_1 = \epsilon + \epsilon^8$, $p_2 = \epsilon^2 + \epsilon^7$, $p_3 = \epsilon^4 + \epsilon^5$ are periods, then

$$b_1 = p_2 - p_3, \quad b_2 = p_3 - p_1, \quad b_3 = p_1 - p_2.$$

The equation for a gives

$$a_1 = 2(p_3 - p_2), \quad a_2 = 2(p_1 - p_3), \quad a_3 = 2(p_2 - p_1).$$

By forming a table of multiplication for the periods

$$p_1^2 = 2 + p_2, \qquad p_2^2 = 2 + p_3, \qquad p_3^2 = 2 + p_1,$$

$$p_2p_3 = -1 + p_2, \qquad p_3p_1 = -1 + p_3, \qquad p_1p_2 = -1 + p_1,$$

we find that the pairs (a_1, b_3), (a_2, b_1), (a_3, b_2) satisfy the second and third equation. In the field of ϵ the equation for φ, and therefore that for f, is completely deducible. This field is however not the minimum field. The field determined by any root of the equation for b is the minimum; for all the b's are rationally expressible in terms of any one of them. The a's are mere multiples of the b's. The b equation has the group G_3 and is its own Galois resolvent.

Further examples of the monodromie group being a subgroup can be made by the device of setting the discriminantal factor which gives cycles not in the subgroup equal to a fixed rational, or working on the subgroup form and keeping algebraic rationality while destroying the arithmetic rationality.

For G_4 under G_8, for instance, using the three parameter form and writing

$$\rho = p^2 - \alpha, \qquad t = p + \frac{c^2}{2(p^2 - \alpha)},$$

the factor Δ_1 of the discriminant which gives rise to the single transposition becomes $4\alpha\rho^2$. If now α is a fixed non square, the group does not reduce but the monodromie group does and to G_4 invariant under G_8.

Here

$$b = p - \frac{c^2}{2(p^2 - \alpha)}, \qquad d = \frac{c^4 + 4pc^2(p^2 - \alpha) + 4\alpha(p^2 - \alpha)^2}{16(p^2 - \alpha)^2}.$$

For $c = 0$, $\alpha = 2$, the equation is

$$x^4 + px^2 + \tfrac{1}{2} = 0,$$

and the monodromie is obvious from the real graph. The adjunction of $\sqrt{\alpha}$ reduces the group to the monodromie group.

For the case of C_4 under G_8, we notice that the C_4 function λ is connected with the G_8 form in this way,

$$\lambda^2 = \frac{\Delta_1}{\rho}.$$

If Δ_1/ρ is a square, the group is C_4. If Δ_1 is a square the group is G_4, while if both these forms are squares the group is G_2 and reduction occurs.

If in the form for G_8 we write

$$\Delta_1 = k\rho^3\varphi^2, \qquad c = \gamma\rho, \qquad t = h\rho,$$

there results for the new form

$$b = \frac{(h + \gamma^2)(4 + k\varphi^2)}{(\gamma^2 - 2h)^2}, \qquad d = \frac{[h^2(4 + k\varphi^2) - (\gamma^2 - 2h)^2](4 + k\varphi^2)}{4(\gamma^2 - 2h)^4}.$$

If k is a non square fixed rational, the monodromie group is C_4 and the group G_8. Adjunction of $\sqrt{k}$ reduces the group to C_4.

For an example of H_4 under G_8, using the two-parameter form for G_8, we write

$$\epsilon = -\frac{2(\kappa^2 - 2\lambda - 1)}{\lambda^2}, \qquad \alpha = -\frac{2}{\lambda^2 - 2\lambda - 1}$$

and $x = \xi/\lambda$. The equation becomes

$$\xi^4 + 2(\lambda - 1)^2\xi^2 + 8\lambda\xi + (\lambda^4 - 4\lambda^3 + 6\lambda^2 + 4\lambda + 1) = 0.$$

This is derived from two circles

$$(x - a)^2 + y^2 - 2ay = 0,$$

$$(x - b)^2 + y^2 - 2by = 0,$$

where

$$a = 1 + i, \qquad b = 1 - i, \qquad y = \lambda.$$

The monodromie is evident from the geometry of the real case. The adjunction of i reduces the group to H_4.

NOTE ON THE EXISTENCE OF A MINIMUM OF $\int_{x_0y_0}^{x_1y_1} Pdx + Qdy$.

By Elijah Swift.

In attempting to apply the principles of calculus of variations to the integral

$$J = \int_{x_0y_0}^{x_1y_1} \left(P(xy) + y'Q(xy) \right) dx,$$

Euler's differential equation degenerates into the finite equation

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \tag{1}$$

This case is usually dismissed with the remark that if (1) is an identity the integral is independent of the path, while if it is not an identity the given points (x_0y_0), (x_1y_1) will not lie on the curve defined by it. It is the purpose of this note to find the conditions for a minimum (maximum) in the case that the end points actually lie on this curve. Call the curve defined by (1), C, and assume that all the second partial derivatives of P and Q exist and are continuous in a neighborhood of C, to use Bolza's notation, that P and Q are of class C'' in this neighborhood.

If we form the second variation of J taken along C and apply to it Legendre's transformation, we obtain it in the form

$$\delta^2 J = \frac{\epsilon^2}{2} \int_{x_0}^{x_1} \eta^2 \left\{ \frac{\partial (P_y - Q_x)}{\partial y} \right\} dx;$$

so that evidently it is necessary for a minimum that

$$\frac{\partial (P_y - Q_x)}{\partial y} \geqq 0$$

along C.

If the stronger condition is satisfied, C has no tangent parallel to the Y-axis in the interval under consideration, and the function $P_y - Q_x$, which vanishes along the curve C, is negative below and positive above it. I shall assume that this latter is the case. Then C actually makes J a minimum.

To prove this we have merely to draw any other curve $\bar{C}$ connecting 0 and 1 and lying in the neighborhood of C. For convenience assume it lies entirely above C. The extension to other cases is immediate. Consider the line integral $\int (P + y'Q)dx$ taken along the contour formed by the

curves C and $\bar{C}$ in positive sense. By Green's theorem, which applies under the above assumptions,

$$\int_{C,\bar{C}}(P + y'Q)dx = \int\int\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy,$$

where the latter integral is to be taken over the area bounded by C and $\bar{C}$. But this latter integral is negative since $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is negative above C according to our assumptions. Then $J_C - J_{\bar{C}} < 0$ or $J_C < J_{\bar{C}}$, which we wished to prove.

For example,* the work done by a force whose components are $F_x = ky^2$, $F_y = kxy$, is

$$W = k\int_{x_0y_0}^{x_1y_1}\{y^2 + xyy'\}dx.$$

Here

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = y, \qquad \frac{\partial}{\partial y}(P_y - Q_x) = 1 > 0.$$

Hence the above force will do the least work on a particle describing a path from $(a, 0)$ to $(b, 0)$, $b > a$, if the particle moves along the X-axis.

PRINCETON, N. J.

* Smith and Longley, Theoretical Mechanics, p. 112, Ex. 4.

CONTINUANT EXPRESSIONS FOR $\sqrt{a^2+b}$ AND $(\sqrt{a^2+b}+a)^n$.

By L. H. Rice.

The line of reasoning employed in a paper by Muir* is applicable to a more general expression than that with which his paper was concerned.

We shall first show that *if the positive integral powers of $\sqrt{a^2+b}+a$ be taken, and the expansion of each be separated into two parts, rational and irrational, then the ratio of the rational portion to the coefficient of $\sqrt{a^2+b}$ approaches as a limit $\sqrt{a^2+b}$ or $-\sqrt{a^2+b}$, as the index of the power approaches infinity, according as a is positive or negative.*

Manifestly,

$$(\sqrt{a^2+b}+a)^n = \frac{(\sqrt{a^2+b}+a)^n + (-1)^{n-1}(\sqrt{a^2+b}-a)^n}{2}$$
$$+\frac{(\sqrt{a^2+b}+a)^n + (-1)^n(\sqrt{a^2+b}-a)^n}{2};$$

and in this expression the first fraction always contains $\sqrt{a^2+b}$ as a factor, while the second fraction is always rational. Consequently we write

$$(\sqrt{a^2+b}+a)^n = \frac{(\sqrt{a^2+b}+a)^n + (-1)^{n-1}(\sqrt{a^2+b}-a)^n}{2\sqrt{a^2+b}}\sqrt{a^2+b}$$
$$+\frac{(\sqrt{a^2+b}+a)^n + (-1)^n(\sqrt{a^2+b}-a)^n}{2}, \tag{1}$$

thereby separating the expansion as specified. The nth convergent,

$$\frac{(\sqrt{a^2+b}+a)^n + (-1)^n(\sqrt{a^2+b}-a)^n}{(\sqrt{a^2+b}+a)^n + (-1)^{n-1}(\sqrt{a^2+b}-a)^n}\sqrt{a^2+b},$$

may be put into either of the forms

* Muir, Thos., "Note on a theorem regarding a series of convergents to the roots of a number," Proc. Roy. Soc. Edin., vol. XIX, p. 15.

$$\frac{1+(-1)^n\left(\frac{\sqrt{a^2+b}-a}{\sqrt{a^2+b}+a}\right)^n}{1+(-1)^{n-1}\left(\frac{\sqrt{a^2+b}-a}{\sqrt{a^2+b}+a}\right)^n}\sqrt{a^2+b}, \qquad \frac{\left(\frac{\sqrt{a^2+b}+a}{\sqrt{a^2+b}-a}\right)^n+(-1)^n}{\left(\frac{\sqrt{a^2+b}+a}{\sqrt{a^2+b}-a}\right)^n+(-1)^{n-1}}\sqrt{a^2+b}.$$

If a is positive, the first form shows that the limit is $\sqrt{a^2+b}$; if a is negative, the second form shows that the limit is $-\sqrt{a^2+b}$.

Ramus,* in 1856, obtained a result which we may express in the form

$$\begin{vmatrix} a & b & & \\ -1 & a & b & \\ & -1 & a & b \\ & & \cdot & \cdot & \cdot \end{vmatrix}_{n-1} = \frac{1}{\sqrt{a^2+4b}}\left[\left(\frac{a+\sqrt{a^2+4b}}{2}\right)^n-\left(\frac{a-\sqrt{a^2+4b}}{2}\right)^n\right],$$

or, replacing a by $2a$, and making a further obvious modification,

$$\begin{vmatrix} 2a & b & & \\ -1 & 2a & b & \\ & -1 & 2a & b \\ & & \cdot & \cdot & \cdot \end{vmatrix}_{n-1} = \frac{(\sqrt{a^2+b}+a)^n+(-1)^{n-1}(\sqrt{a^2+b}-a)^n}{2\sqrt{a^2+b}}.$$

We also have, from the properties of continuants,

$$\begin{vmatrix} a & b & & \\ -1 & 2a & b & \\ & -1 & 2a & b \\ & & \cdot & \cdot & \cdot \end{vmatrix}_{n} = a\begin{vmatrix} 2a & b & \\ -1 & 2a & b \\ & \cdot & \cdot & \cdot \end{vmatrix}_{n-1} + b\begin{vmatrix} 2a & b & \\ -1 & 2a & b \\ & \cdot & \cdot & \cdot \end{vmatrix}_{n-2},$$

whence, since $b=(\sqrt{a^2+b}+a)(\sqrt{a^2+b}-a)$, the continuant on the left of the last equation is equal to

$$\frac{\begin{array}{r} a(\sqrt{a^2+b}+a)^n+(-1)^{n-1}a(\sqrt{a^2+b}-a)^n+(\sqrt{a^2+b}-a)(\sqrt{a^2+b}+a)^n \\ +(-1)^{n-2}(\sqrt{a^2+b}+a)(\sqrt{a^2+b}-a)^n \end{array}}{2\sqrt{a^2+b}}$$

$$=\frac{(\sqrt{a^2+b}+a)^n+(-1)^n(\sqrt{a^2+b}-a)^n}{2},$$

* Ramus, C., "Determinanternes Anvendelse til at bestemme hoven for de convergerende Brøker." Oversigt . . . danske Vidensk. Selsk. Forhandl. . . . Kjøbenhavn, pp. 106–119. Muir's Theory of Determinants, vol. II, p. 427.

which is the rational term in (1). We may therefore rewrite (1) in the forms

$$(\sqrt{a^2+b}+a)^n = \begin{vmatrix} 2a & b & & \\ -1 & 2a & b & \\ & & \cdot & \cdot & \cdot \end{vmatrix}_{n-1} \sqrt{a^2+b} + \begin{vmatrix} a & b & & \\ -1 & 2a & b & \\ & -1 & 2a & b \\ & & \cdot & \cdot & \cdot \end{vmatrix}_{n}$$

$$= \begin{vmatrix} 1 & \sqrt{a^2+b} & & & \\ -1 & a & b & & \\ & -1 & 2a & b & \\ & & -1 & 2a & b & \cdot \\ & & & \cdot & \cdot & \cdot \end{vmatrix}_{n+1} . \qquad (A)$$

We found that

$$\sqrt{a^2+b} = \begin{vmatrix} a & b & & \\ -1 & 2a & b & \\ & -1 & 2a & b \\ & & \cdot & \cdot & \cdot \end{vmatrix} \div \begin{vmatrix} 2a & b & \\ -1 & 2a & b \\ & \cdot & \cdot & \cdot \end{vmatrix} \qquad (a > 0),$$

and

$$\sqrt{a^2+b} = -\begin{vmatrix} a & b & & \\ -1 & 2a & b & \\ & -1 & 2a & b \\ & & \cdot & \cdot & \cdot \end{vmatrix} \div \begin{vmatrix} 2a & b & \\ -1 & 2a & b \\ & \cdot & \cdot & \cdot \end{vmatrix} \qquad (a < 0).$$

By the rule for changing the signs of the principal diagonal elements of a continuant, the latter equation becomes

$$\sqrt{a^2+b} = \begin{vmatrix} |a| & b & & \\ -1 & 2|a| & b & \\ & -1 & 2|a| & b \\ & & \cdot & \cdot & \cdot \end{vmatrix} \div \begin{vmatrix} 2|a| & b & \\ -1 & 2|a| & b \\ & \cdot & \cdot & \cdot \end{vmatrix},$$

which holds for both positive and negative values of a. Hence we have, finally,

$$\sqrt{a^2+b} = |a| + \frac{b}{2|a|+}\,\frac{b}{2|a|+}\cdots. \qquad (B)$$

A part of this result will be seen to furnish a proof of the truth of an equation put forth as a problem in Chrystal's Algebra, Part II, Exs. XXXI, No. (9).

In the proof leading up to equation (B) it is a necessary condition, in case b is negative, that $a^2 > |b|$.

Syracuse University,
October 29, 1912

ON THE UNIFORMIZATION OF ALGEBRAIC FUNCTIONS.

BY WILLIAM F. OSGOOD.

In his investigations on the uniformization of algebraic functions by means of automorphic functions with domain of definition T, *Koebe** has solved two central problems.

I. THE DOMAIN T, BOUNDED BY A CIRCLE.

THEOREM I. *Let w be an algebraic function of z of deficiency $p > 1$. Then there exist two automorphic functions, $\varphi(t)$, $\psi(t)$, whose domain of definition T is the interior of the unit circle, $|t| < 1$, and such that the algebraic configuration $\mathfrak{A}$ whose points are (w, z) is represented by the pair of equations:*

$$z = \varphi(t), \qquad w = \psi(t).$$

To each point t_0 within T corresponds a single point (w_0, z_0) of $\mathfrak{A}$. Conversely, to each point (w_0, z_0) of $\mathfrak{A}$ correspond points $t_0, t_0', \cdots$, interior to T and having as their points of condensation the boundary of T.

Finally, to a certain neighborhood of (w_0, z_0) corresponds a region of T including t_0 in its interior and such that the relation between the points of the two regions in question is one-to-one.

II. THE DOMAIN T, BOUNDED BY THE POINTS OF A DISCRETE SET.

A set of points in the t-plane shall be said to be *discrete* when it satisfies the following conditions. Let $t = a$ be any finite point of the set, and let a circle of arbitrarily small radius be drawn about a as center. Then it shall be possible to draw a simple closed curve inside this circle, which encloses the point a and does not go through either a point of the set or a cluster point of the set.

It is clear that the points of the plane that remain after the points of a discrete set and their limiting points have been removed constitute a single continuum; and, furthermore, that this latter region extends into every neighborhood of each point of the plane.

In the case of the second theorem T is a region whose boundary consists of a discrete set of points.

* Koebe, Mathematische Annalen, vol. 67 (1909), p. 145 and vol. 69 (1910), p. 1. These papers were preceded by a series of notes in the Göttinger Nachrichten beginning in 1907.

THEOREM II. *Let w be an algebraic function of z of deficiency $p > 0$. Then there exist two automorphic functions,* $\varphi(t)$, $\psi(t)$, whose domain of definition T is bounded by the points of a discrete set, and which are such that the algebraic configuration $\mathfrak{A}$ whose points are (w, z) is represented by the pair of equations*

$$z = \varphi(t), \qquad w = \psi(t)$$

in the same sense as in Theorem I.

Koebe has collected his earlier results and given a systematic treatment of Theorem I and related problems in the first of the *Annalen* papers above cited. To cull from the eighty pages of this article that which is essential in the methods is a task of some labor, and furthermore the proofs admit of simplification. In the second edition of my *Funktionentheorie*, vol. 1, I have given what seems to me to be a simple and complete treatment of this problem. The treatment is also typical for the other problems of *Koebe's* first article. It is the purpose of the present paper to help the reader to do the same thing for Theorem II and the related problems.

§ 1. The Surfaces Φ_n and $\Phi = \lim\limits_{n=\infty} \Phi_n$.

Consider an algebraic Riemann's surface F,—i. e., a Riemann's surface corresponding to an algebraic function,—of deficiency $p > 0$. It is possible to draw p loop cuts in it† and still not have it fall apart. This is, however, the maximum number of such cuts that can be drawn in F. The surface thus cut, F_1, is called a surface of *planar character* (*schlichtartig*), and this term will be applied to any surface of two sides which is cut in two by every loop cut. The loop cuts may be taken as analytic curves lying in the finite region.‡

Suppose we have an unlimited number of duplicates of F_1. Let a surface Φ_2 be constructed by extending F_1 by $2p$ duplicates, one being joined to F_1 along the opposite side of each bank of each of the p cuts. The resulting surface is also of planar character, and like F_1 it has a finite number of

* By an *automorphic function* shall be understood in this paper a *single-valued* automorphic function.

† When a simultaneous system of loop cuts is drawn in a surface, it is understood that they are not to intersect one another, and that each loop cut is a simple curve on the surface.

‡ A loop cut may be taken, first, as a polygon whose sides are each parallel to one of the coordinate axes. The vertices can now be smoothed off by easement curves, so that the new loop cut has continuous curvature. The coordinates of the latter cut can be represented parametrically by two functions, each of which can be developed into a Fourier's series differentiable term by term, the derivative series converging uniformly. A suitable number of terms from each of these developments being chosen and their sum being set equal to the proper coordinate, the new curve thus defined is a loop cut lying uniformly arbitrarily near to the original cut and being simple and analytic throughout.

leaves and branch points, and is bounded by a finite number of closed analytic curves. Moreover, F_1, inclusive of its boundary, lies within Φ_2.

Next, construct a surface Φ_3 out of Φ_2 in a precisely similar manner by extending Φ_2 across each of its boundaries by means of further duplicates of F_1. By repeating the process, an unlimited sequence of surfaces

$$\Phi_1 = F_1, \quad \Phi_2, \quad \Phi_3, \quad \cdots$$

is obtained, each with the above mentioned characteristics of Φ_1 and Φ_2. The limiting surface shall be denoted by Φ.

§ 2. The Map of Φ_n on a Plane Region.

We proceed next to the consideration of the following theorem.

The region Φ_n can be mapped by a function

$$t = F_n(z)$$

in a one-to-one manner and continuously, and in general conformally, on a region T_n consisting of the extended single-leaved t-plane with the exception of slits. The latter consist of segments of right lines, finite in number and parallel to the axis of reals, and they correspond to the boundary curves of Φ_n. Furthermore, an ordinary interior point O of Φ_1 having been chosen at pleasure and the origin $z = 0$ having been transformed to it, the neighborhood of O in its sheet of Φ_1 is mapped on the neighborhood of the point $t = \infty$ as follows:

$$t = F_n(z) = \frac{1}{z} + \omega_n(z),$$

where $\omega_n(z)$ remains finite in the neighborhood of O.

Finally, one of these segments shall pass through the point $t = 0$.*

To prove this theorem, we begin by constructing a function η_0 which shall be single-valued and continuous and in general harmonic on the surface Φ_n, except at the point O, where

$$(1) \qquad \eta_0 = -\frac{\sin\theta}{r} + \mu(x, y),$$

μ remaining finite at O. On the boundary C of Φ_n, η_0 shall vanish:†

$$\eta_0\big|_C = 0.$$

* A proof of this theorem is sketched by Koebe in the second of the *Annalen* papers above cited, § 13. The problem has also been treated by Cecioni, Rend. Circ. Mat. Palermo, vol. 25 (1908), p. 1; Hilbert, Göttinger Nachrichten, 1909, p. 314, and Courant, Göttingen Dissertation, Math. Annalen, vol. 72 (1912), p. 517.

† The existence of such a function is established by the method of successive approximations as applied by Schwarz and Neumann. For the details of the proof cf. Osgood, Lehrbuch der Funktionentheorie, vol. 1, 2d ed., 1912, Ch. 14, § 9, in particular, footnote, p. 717.

Let ξ_0 be the negative of the function conjugate to η_0. Then, in the neighborhood of O,

$$(2) \qquad \xi_0 = \frac{\cos\theta}{r} + \lambda(x, y),$$

λ remaining finite there. In general ξ_0 will be multiple-valued on Φ_n. We proceed to replace η_0 by such a function η_n that its conjugate will be single-valued.

Denote the curves that form the boundary of Φ_n by $C_1, \cdots, C_N$. Let $v_1, \cdots, v_N$ be functions single-valued and continuous in Φ_n, and harmonic in the ordinary points of Φ_n, and let

$$v_j\big|_{C_j} = 1; \qquad v_j\big|_{C_k} = 0, \qquad j \neq k.$$

Let $u_1, \cdots, u_N$ be the negatives of the functions which are conjugate respectively to $v_1, \cdots, v_N$. Denote the moduli of periodicity of $u_1, \cdots, u_N$ along (not across) the curve C_j, when that curve is described in the positive sense, by

$$(3) \qquad \omega_1^{(j)}, \quad \cdots, \quad \omega_N^{(j)}.$$

Let $c_1, \cdots, c_N$ be N arbitrary constants, and form the functions

$$u = c_1 u_1 + \cdots + c_N u_N,$$

$$v = c_1 v_1 + \cdots + c_N v_N.$$

The modulus of periodicity of u along the curve C_j is

$$(4) \qquad c_1 \omega_1^{(j)} + c_2 \omega_2^{(j)} + \cdots + c_N \omega_N^{(j)}.$$

Some of these quantities may vanish, but not all of them, except when all the c's are 0. For, if they did, u would be single-valued. Consider the curve

$$\Gamma: \qquad v = c \neq c_j, \qquad j = 1, 2, \cdots N.$$

Here, Γ is a regular closed curve* on Φ_n, and since Φ_n is of planar character, Γ divides it into two or more pieces. Hence a connected region of Φ_n has as one of its boundaries a part or the whole of Γ, and in this region $v - c$ is always positive, or else always negative. From this it follows that $\partial v/\partial \nu$, where ν denotes the inner normal, does not change sign along the boundary in question, and it cannot vanish along any arc of this boundary, however short.† But the change in the conjugate function $-u$ is given

* The discussion of the isothermals of the Green's function, Funktionentheorie, Ch. 13, § 7, applies to the present locus.

† Funktionentheorie, p. 665.

precisely by the integral

$$-\int \frac{\partial v}{\partial \nu} ds$$

taken over the boundary in question, and here is a contradiction.

As a consequence of the foregoing we have the relation:

$$\begin{vmatrix} \omega_1^{(1)} & \cdots & \omega_N^{(1)} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \omega_1^{(N)} & \cdots & \omega_N^{(N)} \end{vmatrix} \neq 0.$$

Returning now to the functions η_0, ξ_0, let us form the functions

$$\xi_n = \xi_0 + c_1 u_1 + \cdots + c_N u_N + a,$$

$$\eta_n = \eta_0 + c_1 v_1 + \cdots + c_N v_N + b.$$

We perceive that it is always possible so to determine the c's that all the moduli of periodicity of ξ_n shall vanish, and thus ξ_n will be a single-valued function on the surface Φ_n. Let this be done, and furthermore let a, b be so chosen that ξ_n, η_n both vanish in a given boundary point of Φ_n. We note that

$$\eta_n\big|_{C_j} = c_j + b = c_j', \qquad j = 1, 2, \cdots, N.$$

Finally, the function

$$t = \xi_n + i\eta_n = F_n(z) \tag{5}$$

yields the desired map. For, consider the curve

$$K: \qquad \eta_n = c \neq c_j', \qquad j = 1, 2, \cdots, N.$$

This curve, as in the case of Γ above, is seen to be a regular closed curve on the surface, and it passes through O, having there a single branch. Since Φ_n is of planar character, it is cut in two by K. Moreover, K is simple on the surface. For otherwise Φ_n would be cut into more than two pieces by K, and one of these pieces, then, would not abut on O. In this piece, $\eta_n - c$ would be always positive, or else always negative, and thus we are led to a similar situation and contradiction to that on which was based the proof above, that not all the quantities (4) vanish.

Since the value of ξ_n along K is given by the integral

$$\xi_n = \int \frac{\partial \eta_n}{\partial \nu} ds + \text{const.}$$

extended along a variable arc of K, the proper normal ν being chosen, and since $\partial\eta_n/\partial\nu$ does not change sign along K or vanish along any arc of K, it follows, with reference to the relations (1) and (2), that the points of K

are transformed by (5) in a one-to-one manner and continuously into the points of the right line of the $t = \xi + \eta i$-plane:

$$\eta = c.$$

Thus a one-to-one relation is established between all the points of the surface Φ_n for which $\eta_n \neq c_j', j = 1, 2, \cdots, N$, and the points of the extended t-plane exclusive of the N parallels to the axis of reals, $\eta = c_j'$.

To deal with the excepted points let P be an interior point of Φ_n not a branch point or a point ∞, in which $\eta_n = c_j'$. The function (5) maps the neighborhood of P either on a single-leaved region of the t-plane or on the neighborhood of a branch point. From the foregoing result the latter alternative is seen to be impossible. Hence the points of the curve $\eta_n = c_j'$ which lie in the neighborhood of P go over in a one-to-one manner into the points of a segment of the line $\eta = c_j'$. In particular, then, it follows that $\partial\eta_n/\partial x$ and $\partial\eta_n/\partial y$ never vanish simultaneously in a point, P,—or in fact in any finite point of Φ_n. Hence the interior points of Φ_n which lie on the curves $\eta_n = c_j'$ go over in a one-to-one manner into the points of segments of the lines $\eta = c_j'$.

This completes the proof for interior points. It is easily seen that the points of the boundary curves C_j go over in a one-to-two manner into segments of the lines $\eta = c_j'$.

§ 3. The Map of Φ on a Single-Leaved Region.

Let

$$f_n(z) = \frac{1}{F_n(z)}, \qquad z \neq 0; \qquad f_n(0) = 0,$$

the exceptional point $z = 0$ being merely the point O of a single leaf. Then the function

$$t = f_n(z)$$

maps Φ_n on a single-leaved region R_n of the t-plane in a one-to-one manner and continuously, and in general conformally, each interior point of Φ_n going over into a finite point t. Moreover,

$$f_n(0) = 0, \qquad f_n'(0) = 1.$$

The proof of Theorem II turns on the following theorem.

Let Φ_k be chosen arbitrarily from the regions $\Phi_1, \Phi_2, \cdots$. From the set of functions $f_1(z), f_2(z), \cdots$ a set

$$f_{n_1}(z), \qquad f_{n_2}(z), \qquad \cdots \qquad n_i < n_{i+1},$$

can then be selected which converges uniformly in Φ_k. The indices n_i are

independent of k; and it is, moreover, understood that those functions, if any such exist, for which $n_i < k$ are to be omitted.

The limiting function,

$$f(z) = \lim_{i=\infty} f_{n_i}(z),$$

is uniquely defined at each point of Φ, and by means of it,

$$t = f(z),$$

Φ is mapped on a single-leaved region T of the t-plane as required in Theorem II.

To prove this theorem we begin by showing, in the next paragraph, that the functions $f_n(z)$, $k \leqq n$, remain finite in Φ_k.

§ 4. A Lemma.

Consider the functions

$$t = f(z) \tag{1}$$

which map a given circle $|z| < \rho$ on single-leaved regions S not containing the point $t = \infty$ in their interior (though it may lie on the boundary), and which, furthermore, are such that

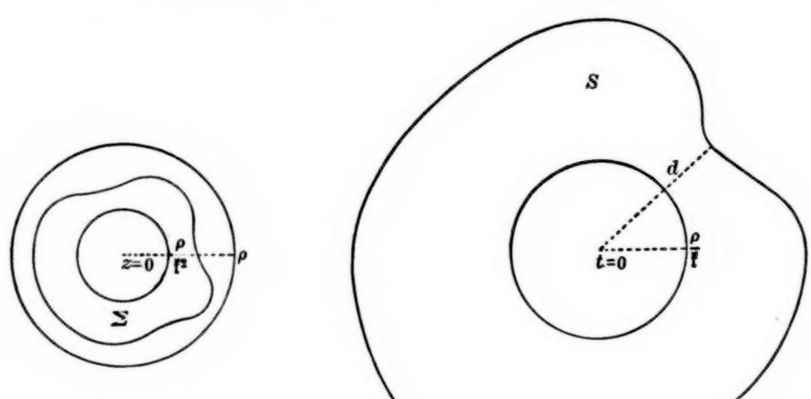
$$f(0) = 0, \qquad f'(0) = 1.$$

These functions remain finite in the circle $|z| \leqq \rho/\mathfrak{k}^2$, where $\mathfrak{k}$ denotes Koebe's constant; in fact*

$$|f(z)| \leqq \frac{\rho}{\mathfrak{k}}, \qquad \textit{where} \qquad |z| \leqq \frac{\rho}{\mathfrak{k}^2}.$$

Moreover,

$$|f'(z)| \leqq 4\,\mathfrak{k}, \qquad \textit{where} \qquad |z| \leqq \frac{\rho}{2\,\mathfrak{k}^2}.$$



* Funktionentheorie, p. 727.

Let d be the distance from the point $t = 0$ to the boundary of S. Then*

$$d \geqq \frac{\rho}{\mathfrak{k}}.$$

Hence the interior of the fixed circle

$$(2) \qquad |t| \leqq \frac{\rho}{\mathfrak{k}}$$

is mapped by each of the functions (1) on a region Σ of the z-plane.

Let δ be the distance from $z = 0$ to the boundary of Σ. Then, by a second application of the above theorem

$$\delta \geqq \frac{\rho}{\mathfrak{k}^2},$$

and hence the fixed circle

$$(3) \qquad |z| \leqq \frac{\rho}{\mathfrak{k}^2}$$

is mapped by each of the functions (1) on a regon of the t-plane lying in the fixed circle (2). Thus the first part of the theorem is established.

To prove the second part, represent $f'(z)$ in the circle

$$(4) \qquad |z| \leqq \frac{\rho}{2\mathfrak{k}^2}$$

by the integral:

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)^2},$$

extended over the circle C: $|z| = \rho/\mathfrak{k}^2$. Here,

$$|f(\zeta)| \leqq \frac{\rho}{\mathfrak{k}}, \qquad |\zeta - z| \geqq \frac{\rho}{2\mathfrak{k}^2}, \qquad |d\zeta| = \frac{\rho}{\mathfrak{k}^2}d\varphi,$$

and from these relations the proof follows at once.

From the results just obtained we can now deduce the following theorem, which includes the foregoing as a special case.

Let Σ be a region spread out over the extended z-plane, having a finite number of leaves and branch points, and bounded by a finite number of regular curves, none of which go through a branch point or the point ∞. Let Σ' be a second region containing Σ in its interior, but having no branch point on the boundary of Σ, and let Σ, Σ' be such that Σ' can be mapped on a single-leaved plane region.

Consider the functions

$$t = f(z)$$

* Funktionentheorie, p. 726, § 12. This theorem, which is due in substance to Koebe, is fundamental both in the investigations referred to at the beginning of this paper and in those in which we are now engaged.

which map Σ' on single-leaved regions of the t-plane not including the point $t = \infty$ in their interior, and which, furthermore, are such that, at an ordinary interior point O, $z = 0$, of Σ

$$f(0) = 0, \qquad f'(0) = 1.$$

These functions $f(z)$ remain finite in Σ.

Let the neighborhoods of the points $z = \infty$ and of the branch points be removed from Σ, the new region $\bar{\Sigma}$, which is finite and without branch points, having as its additional boundaries circles. It is evidently sufficient to show that, *in the region $\bar{\Sigma}$, the functions $f(z)$ remain finite.*

It is possible to divide the plane up into a net work of squares, of length h on a side, and such that the various leaves of $\bar{\Sigma}$ are divided by the projection of the boundaries of these squares into a finite number of squares or pieces of squares in the following manner. A circle of radius

$$\rho = 2\sqrt{2}h \cdot 2\mathfrak{k}^2 = 4\sqrt{2}\mathfrak{k}^2 h$$

with its center within or on the boundary of a square which includes interior points of $\bar{\Sigma}$ shall not reach out in its leaf to a branch point or boundary point of Σ'. In a square which contains O in its interior or on its boundary the functions $f(z)$ are seen from the theorem at the beginning of the paragraph to remain finite, and moreover $|f'(z)| \leqq 4\mathfrak{k}$ there.

Let s be any one of the squares in which $f(z)$ and $f'(z)$ remain finite; s_1, an adjacent square, and z_1 a point on the boundary of each. These squares shall be considered as including their boundaries. Then $f(z)$ and $f'(z)$ remain finite in s_1. For, $f'(z_1) \neq 0$, and the function

$$\bar{f}(z) = \frac{f(z) - f(z_1)}{f'(z_1)}$$

satisfies the conditions of the theorem proven at the beginning of the paragraph, the point z_1 corresponding to O.

Since there are only a finite number of squares to be considered, the finiteness of $f(z)$ and $f'(z)$ in their totality is herewith established, and thus the theorem is proven. Incidentally we have obtained the further result: *In the region $\bar{\Sigma}$ the functions $f'(z)$ also remain finite.*

We are now in a position to take the first step in the proof of the theorem of § 3, indicated at the close of that paragraph. Let Σ be taken as the domain Φ_k, and Σ' as a domain Φ_m, $m > k$. The functions $f_n(z)$, $n \geqq m$, are then included in the class of functions $f(z)$ of the theorem of the present paragraph, and hence we see that the functions $f_n(z)$, $n > k$, remain finite in Φ_k.

The next step is to show that from these functions a set $f_{n_i}(z)$ can be selected which converge uniformly in Φ_k.

§ 5. A Theorem in Uniform Convergence.

We will begin by stating a theorem for a real function of a real variable, which contains that which is essential in the theorem that follows later in the paragraph.

Let $f_n(x)$ be a real function of the positive integer n and the real variable x in the finite closed interval $a \leqq x \leqq b$; and let

(a) $f_n(x)$, regarded as a function of x and n, be finite:

$$|f_n(x)| < M, \qquad a \leqq x \leqq b, \qquad n = 1, 2, 3, \cdots,$$

M being a positive constant;

(b) let the difference quotient also remain finite

$$\left|\frac{f_n(x') - f_n(x'')}{x' - x''}\right| < M',$$

where x' and x'' are any two distinct points of the above interval, and n is arbitrary; M' being a positive constant.

Then it is possible to choose from the functions $f_n(x)$ a set

$$f_{n_1}(x), \quad f_{n_2}(x), \quad \cdots,$$

*which converges uniformly in the above interval (a, b).**

From (b) it follows that $f_n(x)$ is a continuous function of x in the closed interval (a, b). Moreover, Condition (b) will always be fulfilled when $f_n(x)$ possesses a derivative which, regarded as a function of x and n, remains finite.

To prove the theorem, let $a_1, a_2, \cdots$ be a set of distinct points of (a, b) everywhere dense in this interval. For convenience, take them as the points of division when the interval is divided into 2^k equal intervals, $k = 1, 2, 3, \cdots$.

From the set of functions $f_n(x)$ it is possible to choose a set which we will denote by

$$(1) \qquad f_{1,1}(x), \quad f_{1,2}(x), \quad \cdots,$$

and which converge for the value of the argument $x = a_1$.

From the set (1) of functions we can now select a set

$$(2) \qquad f_{2,1}(x), \quad f_{2,2}(x), \quad \cdots$$

* The theorem and its proof are closely related to the investigations of de la Vallée-Poussin on differential equations, 1892; Mém. Acad. Belgique, 8°, vol. 47 (1892–95). Cf. further Townsend, Göttingen Dissertation, 1900, p. 34. In the form given below for harmonic functions it was used in part by Hilbert, Über das Dirichletsche Prinzip, Festschrift, 150th anniversary of the Göttingen Academy, 1901, p. 8. Koebe refers to Ascoli, 1883; Math. Annalen, vol. 69 (1910), p. 71.

which converge for $x = a_2$. From this set is selected a set converging for $x = a_3$, etc.

We are thus let to a double array of functions

$$\begin{array}{ccc} f_{1,1}(x), & f_{1,2}(x), & \cdots \\ f_{2,1}(x), & f_{2,2}(x), & \cdots \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \end{array}$$

from which we can select in a great variety of ways a set converging in each of the points $a_1, a_2, \cdots$. For example, it would suffice to choose the functions of the principal diagonal of the array:

$$(3) \qquad f_{1,1}(x), \quad f_{2,2}(x), \quad \cdots.$$

So much without the use of Condition (b). It is now an easy matter to show by the aid of this condition that the set (3) converges uniformly in the closed interval (a, b), and this completes the proof.

We proceed now to a second theorem, the proof of which is similar to that which has just been given, and which contains in substance the final result we desire.

Let S be an arbitrary Riemann's surface and let

$$u_1(x, y), \quad u_2(x, y), \quad \cdots$$

be a set of functions single-valued and continuous within S, and harmonic in the ordinary points of S. Furthermore, let $u_n(x, y)$, regarded as a function of the point (x, y) of S and the positive integer n, be finite:

$$|u_n(x, y)| < M,$$

where M is a constant. Then it is possible to choose from the above set of functions a set

$$u_{n_1}(x, y), \quad u_{n_2}(x, y), \quad \cdots$$

which converges at all interior points of S, and which, moreover, converges uniformly in any preassigned subregion Σ which together with its boundary lies within S.

Let (a_1, b_1) (a_2, b_2), $\cdots$ be a set of distinct ordinary interior points of S everywhere dense in S. From the set of functions $u_n(x, y)$ it is possible to choose a set which we will denote by

$$u_{1,1}(x, y), \quad u_{1,2}(x, y), \quad \cdots$$

and which converges for the point (a_1, b_1).

From this set we can now select a set

$$u_{2,1}(x, y), \quad u_{2,2}(x, y), \quad \cdots$$

which converges for the point (a_2, b_2). And so on.

We are thus led to a double array:

$$\begin{array}{ccc} u_{1,1}(x, y), & u_{1,2}(x, y), & \cdots \\ u_{2,1}(x, y), & u_{2,2}(x, y), & \cdots \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \end{array}$$

The principal diagonal of this array will yield a set of functions $u_{i,i}(x, y) = u_{n_i}(x, y)$ which converge at every point (a_l, b_l) of the above set.

Let (x_0, y_0) be an ordinary interior point of S. With this point as center describe a circle C which contains in its interior and on its boundary only ordinary interior points of S, exclusive of branch points. In the interior of this circle the value of $u_n(x, y)$ for the points of the sheet of S in which C lies is given by Poisson's integral:

$$u_n(x, y) = \frac{1}{2\pi}\int_0^{2\pi} U_n \frac{(a^2 - r^2)\, d\psi}{a^2 - 2ar\cos(\theta - \psi) + r^2},$$

where U_n denotes the value of u_n on the circumferene of C, and (r, θ) are the polar coördinates of a point of C referred to (x_0, y_0) as pole.

The partial derivatives of $u_n(x, y)$ at interior points of C are given by differentiating under the sign of integration. On writing down these formulas it becomes evident *that $\partial u_n/\partial x$, $\partial u_n/\partial y$ are finite throughout a circle concentric with C and of smaller radius.*

This property of the functions $u_n(x, y)$ supplies the place of Condition (b) in the earlier theorem, and thus it is easy to prove that the set of functions $u_{n_i}(x, y)$ defined above converges uniformly throughout a suitably restricted neighborhood of the point (x_0, y_0).

The passage from this last result to the uniform convergence in the above region Σ is effected as follows. The boundary of Σ may be assumed to pass through no branch point, since otherwise Σ could be replaced by a more comprehensive region of the same nature and having this property. And now the boundary can be divided up into a finite number of arcs, each lying in a region in which the functions in question converge uniformly. Hence the functions converge uniformly along the boundary, and consequently throughout the interior, of Σ. This completes the proof.

Application to the Functions $f_n(z)$.

We are now in a position to show that from the functions $f_n(z)$ of § 3 a set

$$f_{n_1}(z), \quad f_{n_2}(z), \quad \cdots$$

can be selected converging uniformly in the arbitrary region Φ_k. In fact, we have but to write

$$f_n(z) = u_n(x, y) + iv_n(x, y), \qquad n > 2,$$

and apply the foregoing theorem to $u_n(x, y)$, Φ_2 and Φ_1 being taken as the regions S and Σ. We are thus led to a first set of functions

$$(4) \qquad f_{1,1}(z), \quad f_{1,2}(z), \quad \cdots,$$

converging uniformly in Φ_1.

We now make a second application of the theorem, taking as our functions the set (4) (exclusive of $f_3(z)$ if it occurs there) and as our regions S and Σ the regions Φ_3 and Φ_2. We are thus led to a second set of functions

$$(5) \qquad f_{2,1}(z), \quad f_{2,2}(z), \quad \cdots,$$

converging uniformly in Φ_2.

Repeating the process for the regions Φ_4 and Φ_3, and the functions (5) (exclusive of $f_4(z)$ if it occurs there), we obtain a third set, and so on. Thus we have a double array of functions

$$\begin{array}{ccc} f_{1,1}(z), & f_{1,2}(z), & \cdots, \\ f_{2,1}(z), & f_{2,2}(z), & \cdots, \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot \end{array}$$

If, now, from these we choose, say, the functions of the principal diagonal and set

$$f_{i,i}(z) = f_{n_i}(z),$$

the functions

$$(6) \qquad f_{n_1}(z), \quad f_{n_2}(z), \quad \cdots$$

will form a set such as is desired. The indices n_i are independent of k, and the functions (6),—a fixed number of terms at the beginning having been suppressed if necessary,—converge uniformly in any given Φ_k.

We are thus led to a limiting function

$$\lim_{i=\infty} f_{n_i}(z) = f(z),$$

single-valued in Φ, and the next step consists in showing that by it:

$$t = f(z),$$

Φ is mapped on a single-leaved domain T of the t-plane.

§ 6. The Map Defined by $t = f(z)$.

The desired proof is furnished at once by a theorem of Hurwitz's* which says that if $\varphi(z, n)$ is analytic in a region S and continuous on the boundary, and if $\varphi(z, n)$ converges uniformly in the closed region S, the limiting function $\varphi(z)$ not vanishing at any point of the boundary, then the number of zeros of $\varphi(z)$ in S is the same as the number of zeros of $\varphi(z, n)$ in S for all values of n from a definite point on: $n \geqq m$.

Suppose, then, that $f(z)$ were to take on the same value A in two distinct points of Φ, P and Q. Let S be a region which together with its boundary lies within Φ, contains the points P and Q in its interior, and is such that $f(z)$ does not take on the value A on its boundary. Then $\varphi(z, i) = f_{n_i}(z) - A$ satisfies the conditions of Hurwitz's theorem.

As soon, however, as i is large enough for the region Φ_{n_i} to enclose S, this latter function has at most a single zero in S, since the function $t = f_{n_i}(z)$ maps Φ_{n_i} on a single-leaved region. This proves the assertion that Φ is mapped by the function $t = f(z)$ on a single-leaved region T of the t-plane.

Let

$$z = \varphi(t)$$

be the function defined by this map, i. e., the inverse of the function $t = f(z)$. Then $\varphi(t)$ is single valued in T, and the function w of § 1 goe. over into a function $w = \psi(t)$ likewise single-valued and analytic in Ts Thus the given algebraic configuration has been uniformized, and it remains merely to show that the functions $\varphi(t)$, $\psi(t)$ are automorphic, and that the boundary of T consists of a discrete set of points.

§ 7. The Function $\phi(t)$ Automorphic.

It is clear that the surface Φ admits an enumerably infinite group of conformal transformations into itself, the generators of which consist of those transformations obtained by projecting the first leaf of $\Phi_1 = F_1$ on the corresponding leaf of any duplicate of F_1 which forms a part of Φ.

To each of these transformations corresponds a transformation of T into itself which is one-to-one and conformal without exception. We proceed to prove that *each of these transformations is linear.*

It will be convenient to replace the region T by the region **T** obtained from T by the transformation

$$t = \frac{1}{x},$$

* Funktionentheorie, p. 722. Cf. also the application of this theorem in the paragraph cited.

in order that the boundary may lie in the finite region of the plane. Let

$$(1) \qquad \varphi\left(\frac{1}{x}\right) = \omega(x); \quad \text{then} \quad z = \omega(x),$$

and let the transformations of **T** into itself corresponding to those above mentioned be denoted by

$$(2) \qquad x' = \chi_n(x) \qquad\qquad n = 1, 2, \cdots.$$

The initial region $\Phi_1 = F_1$ of Φ is mapped by the function (1) on a region $\mathbf{T}_1$ which includes the point $x = \infty$ in its interior and is bounded by $2p$ simple closed non-intersecting analytic curves exterior to one another. Among the transformations (2) there are p which, together with their inverses, carry $\mathbf{T}_1$ over into $2p$ regions lying respectively in the $2p$ interiors of the curves which bound $\mathbf{T}_1$, each of these regions having with $\mathbf{T}_1$ a common boundary. There are further transformations (2) which carry $\mathbf{T}_1$ into regions lying respectively in the interiors of the curves which form the inner boundaries of the latter regions and having each a boundary in common with one of these regions. And so on indefinitely.

We will denote the number of the inner boundary curves at the end of the n-th step by N, the curves themselves by $C_k^{(n)}$, $k = 1, 2, \cdots, N$, and their respective lengths by $l_k^{(n)}$. Thus for the initial region $n = 1$, $N = 2p$. Let $\mathbf{T}_n$ be the part of **T** exterior to the curves $C_k^{(n)}$, $k = 1, 2, \cdots, N$.

The proof that the transformations (2) are linear depends essentially on the following

LEMMA. *The series*

$$(3) \qquad \sum_{n=1}^{\infty} \sum_{k=1}^{N} [l_k^{(n)}]^2$$

converges.

The proof of this lemma depends in turn on a theorem relating to the amount of deformation in certain conformal maps (*Verzerrungssatz,* Koebe), to which we now turn.

§ 8. The Amount of Deformation in Certain Conformal Maps.

Let Σ' be any single-leaved region of the z-plane not including the point $z = \infty$ in its interior, and let Σ be a region which together with its boundary lies within Σ'. Consider the functions*

$$Z = f(z)$$

* The theorem holds for a multiple-leaved region, provided Σ is taken, like the $\bar{\Sigma}$ of § 4, as a finite subregion without branch points.

which map Σ' conformally on single-leaved regions S' not including the point $Z = \infty$ in their interiors. Then, if z_1 and z_2 be any two distinct points of Σ, the ratio $f'(z_1)/f'(z_2)$ remains finite:

$$\left|\frac{f'(z_1)}{f'(z_2)}\right| < M,$$

where M is a constant.

Let $O: z = 0$, be an interior point of Σ. The function

$$F(z) = \frac{f(z) - f(0)}{f'(0)}$$

satisfies the conditions imposed on $f(z)$ in § 4, the present region Σ, or if necessary a larger one, corresponding to the region Σ of that paragraph, and it follows from that paragraph that $F'(z)$ is finite in Σ. Since

$$\frac{f'(z_1)}{f'(z_2)} = \frac{F'(z_1)}{F'(z_2)},$$

it will be sufficient to show that $|F'(z)|$ has a positive lower limit in Σ.

Suppose this were not the case. Then it would be possible to find a set of functions of the above class, $f_1(z)$, $f_2(z)$, $\cdots$, together with their related functions

$$(1) \qquad F_n(z) = \frac{f_n(z) - f_n(0)}{f_n'(0)},$$

and a set of points $z_1, z_2, \cdots$ in Σ, such that

$$(2) \qquad |F_n'(z_n)| < \epsilon \qquad n \geqq m.$$

The points z_n have at least one point of condensation in Σ, and it will be convenient to take them as having but one, $\lim\limits_{n=\infty} z_n = \bar{z}$.

Let Σ_1 be a finite region lying within Σ' and enclosing Σ in its interior. Then, since $F_n(z)$ remains finite in Σ_1 by § 4, it is possible by the method of § 5 to pick out from the functions (1) a set of functions

$$F_{n_1}(z), \quad F_{n_2}(z), \quad \cdots$$

converging uniformly in Σ_1 and such that the indices n_i are independent of the particular choice of Σ_1. It follows, then, by applying Hurwitz's theorem as in § 6, that the limiting function*

$$F(z) = \lim_{i=\infty} F_{n_i}(z)$$

also corresponds to a function of the class considered in the theorem.

* That this function does not vanish identically is seen from the fact that $F'_{n_i}(0) = 1$.

We now proceed to show that

$$F'(\bar{z}) = 0.$$

From this contradiction follows the truth of the theorem.

Since $F'_{n_i}(z)$ converges uniformly in Σ, we have

$$(3) \qquad |F'_{n_i}(z_{n_i}) - F'(z_{n_i})| < \epsilon, \qquad i \geqq \mu.$$

Moreover, from the continuity of $F'(z)$ it follows that

$$(4) \qquad |F'(z_{n_i}) - F'(\bar{z})| < \epsilon, \qquad i \geqq \mu'.$$

Combining (2), written for $n = n_i$, with (3) and (4), we get

$$|F'(\bar{z})| < 3\epsilon,$$

and this completes the proof.

§ 9. Proof of the Lemma of § 7.

We are now in a position to prove the lemma of § 7. Let $l = l_k^{(1)}$, i. e., let l be the length of one of the bounding curves $C = C_k^{(1)}$ of $\mathbf{T}_1$, and let $\Sigma' = \Sigma'_1$ be a finite strip enclosing C in its interior and such that its reproductions Σ'_n under the transformations of the group (2) of § 7 do not overlap. Let $\Sigma = \Sigma_1$ be a strip lying within Σ' and enclosing C in its interior. Then it and its reproductions Σ_n lie in a finite region of the x-plane.

Let l_n be the length of the image of C when the transformation

$$x' = \chi_n(x)$$

is performed on C. Then we can write with Koebe*

$$l_n^2 = \int_0^l \int_0^l |\chi'_n(z_1)| \cdot |\chi'_n(z_2)| \cdot |dz_1| \cdot |dz_2|.$$

Applying to the integrand the algebraic relation

$$2AB \leqq A^2 + B^2,$$

we have

$$(1) \qquad l_n^2 \leqq \frac{1}{2}\int_0^l \int_0^l \left\{ |\chi'_n(z_1)|^2 + |\chi'_n(z_2)| \right\} |dz_1| \cdot |dz_2|.$$

On the other hand, consider the area f_1 of $\Sigma = \Sigma_1$ and the area f_n of Σ_n. Since the ratio of similitude in the two maps is $|\chi'_n(z)|$, we have

$$f_n = \iint_{\Sigma} |\chi'_n(z)|^2 dS,$$

extended over Σ, and hence

$$f_n = |\chi'_n(\zeta)|^2 f_1,$$

* Math. Annalen, vol. 69, p. 28.

where $|\chi_n'(\zeta)|$ denotes a mean value of the integrand, such a value actually being taken on at a point $z = \zeta$ of Σ.

From the theorem of § 8 it follows that at an arbitrary point z of Σ

$$|\chi_n'(z)| < M|\chi_n'(z')|,$$

where z' denotes any second point of Σ, and hence, in particular,

$$|\chi_n'(z)|^2 < M^2|\chi_n'(\zeta)|^2 = \frac{M^2}{f_1}f_n.$$

Applying this result to the integrand of (1), we get

$$l_n^2 \leqq \frac{M^2 l^2}{f_1}f_n.$$

But

$$\sum_{n=1}^{\infty} f_n,$$

being a series of non-overlapping areas which lie in a finite region of the plane, converges. Hence the series

$$\sum_{n=1}^{\infty} l_n^2$$

converges.

The series (3) of § 7 is the sum of the p series here obtained which correspond to the $2p$ boundaries of $\mathbf{T}_1$ taken in pairs,—a pair consisting of two boundary curves which are carried over into each other by one of the transformations (2) of § 7 and its inverse. This completes the proof of the lemma of § 7.

§ 10. Completion of the Proof that $\phi(t)$ is Automorphic.

Let

(1) $$x' = \chi(x)$$

be an arbitrary transformation of the group (2), § 7, and let

(2) $$x'' = L(x')$$

be a linear transformation that carries the point $x' = \chi(\infty)$ back to the point $x'' = \infty$. Let

(3) $$y = f(x) = L[\chi(x)].$$

Then $f(x)$ has a pole of the first order in the point $x = \infty$, and is analytic everywhere else in $\mathbf{T}$.

Let Γ be a simple closed curve that contains in its interior all the boundary points of $\mathbf{T}$. Let x be an arbitrary point of $\mathbf{T}$ lying within Γ. Choose n so that x lies within $\mathbf{T}_n$, and also that the boundary of $\mathbf{T}_n$ lies within Γ. Then $f(x)$ can be represented by Cauchy's integral formula:

$$f(x) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)d\xi}{\xi - x} + \frac{1}{2\pi i}\sum_{k=1}^{N}\int_{C_k^{(n)}} \frac{f(\xi)d\xi}{\xi - x}.$$

The left hand side of this equation and the first term on the right are independent of n. Hence the second term on the right must also be independent of n. We proceed to show that it is independent of x, too, and that it has, in fact, the value 0.

Let n' be a particular value of n satisfying the above condition, and let D be the distance from x to the boundary of $\mathbf{T}_{n'}$. Then, for $n \geqq n'$ and for an arbitrary point ξ of the boundary of $\mathbf{T}_n$,

$$|\xi - x| \geqq D.$$

Let ξ' be a point of the curve $C = C_k^{(n)}$, and let l be the length of this curve. Then

$$\int_C \frac{f(\xi)d\xi}{\xi - x} = f(\xi')\int_C \frac{d\xi}{\xi - x} + \int_C \frac{f(\xi) - f(\xi')}{\xi - x}d\xi.$$

The first integral on the right hand side vanishes by Cauchy's integral theorem. As regards the second,

$$\left|\int_C \frac{f(\xi) - f(\xi')}{\xi - x}d\xi\right| \leqq \frac{1}{D}\int_C |f(\xi) - f(\xi')| \cdot |d\xi|.$$

Let Δ be the oscillation of $f(\xi)$ along C, i. e., the maximum value of $|f(z_1) - f(z_2)|$ for any two points of C. Then

$$\int_C |f(\xi) - f(\xi')| \cdot |d\xi| \leqq \Delta l \leqq \tfrac{1}{2}(\Delta^2 + l^2).$$

Hence it follows that

$$\left|\sum_{k=1}^{N}\int_{C_k^{(n)}} \frac{f(\xi)d\xi}{\xi - x}\right| \leqq \frac{1}{2D}\sum_{k=1}^{N}[(\Delta_k^{(n)})^2 + (l_k^{(n)})^2],$$

where $\Delta_k^{(n)}$ denotes the oscillation of $f(\xi)$ along $C_k^{(n)}$.

The sum on the right hand side can be made arbitrarily small, since, as will now be shown, it is the general term of a convergent series.

The series

$$\sum_{n=1}^{\infty}\sum_{k=1}^{N}[l_k^{(n)}]^2$$

converges by the lemma of § 7.

Furthermore, the curves $C_k^{(n)}$ are carried by the transformation (2) into curves of the finite plane, and thus

$$l_k'^{(n)} < G l_k^{(n)},$$

where $l_k'^{(n)}$ denotes the length of the image of $l_k^{(n)}$, and G is a constant. Hence

$$(4) \qquad \sum_{n=1}^{\infty} \sum_{k=1}^{N} [l_k'^{(n)}]^2$$

is a convergent series.

Consider now the oscillation of $f(\xi)$ along the curve $C_k^{(n)}$. By means of (1) this curve is carried into a curve $C_{k'}^{(n')}$, and the latter curve is carried by (2) into a curve C' of length $l_{k'}'^{(n')}$. Hence the oscillation of $f(\xi)$ along $C_k^{(n)}$, being equal to the maximum diameter of C', is less than half its length, or

$$\Delta_k^{(n)} < \tfrac{1}{2} l_{k'}'^{(n')}.$$

But the series

$$\sum_{n'} \sum_{k'} [l_{k'}'^{(n')}]^2$$

is made up of the same terms as the series (4). Hence the series

$$\sum_{n=1}^{\infty} \sum_{k=1}^{N} [\Delta_k^{(n)}]^2$$

converges.

We have thus arrived at the following representation of $f(x)$:

$$f(x) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi) d\xi}{\xi - x}.$$

Thus it appears that $f(x)$ can be continued analytically over the whole interior of Γ, and $f(x)$ is, therefore, a linear function of x. Consequently $\chi(x)$ is also a linear function of x, and the automorphic character of the function $\omega(x)$, and with it $\varphi(t)$, is herewith established.

§ 11. The Boundary of T Discrete.

Finally, it remains to show that the boundary of T is a discrete set of points.

This property follows at once for **T**, and hence also for T, from the lemma of § 7, since all the boundary points of **T** are included in the interiors of the curves $C_k^{(n)}$, and the length of each of these curves can be made arbitrarily small by a suitable choice of n.

Incidentally we have obtained the result that the area of the portion of the x-plane exterior to $\mathbf{T}_n$ can be made arbitrarily small by choosing n large enough; for the area enclosed by $C_k^{(n)}$ obviously cannot exceed $[l_k^{(n)}]^2/4\pi$. Thus the boundary of **T** can be enclosed in a finite number of regions whose total area is arbitrarily small.

Harvard University,
October 3, 1912.

MANIFOLDS OF N DIMENSIONS.*

By O. Veblen and J. W. Alexander, II.

Introduction.

1. A complete classification of manifolds from the point of view of analysis situs still remains to be made, althought Betti† and Riemann‡ have shown that with every n-dimensional manifold there may be associated a set of constants

$$B_1, \quad B_2, \quad B_3, \quad \cdots, \quad B_{n-1}$$

which are obtained by generalizing the notion of the connectivity of a surface.

Poincaré has proved§ that any manifold M_n may be completely characterized from a topological point of view by means of certain suitably chosen matrices, and has shown how to derive from these matrices a set of positive integers||

$$P_1, \quad P_2, \quad P_3, \quad \cdots, \quad P_{n-1}$$

which are invariants of M_n. These numbers have been called by him the "Betti numbers" on account of to their close resemblance to the numbers B_i of Betti and Riemann. Poincaré has also used the matrices in deriving his *coefficients of torsion* as well as in discussing the *fundamental group* of a manifold. The numbers P_i determined by a given *two-sided* manifold satisfy two relations, a theorem of duality

$$(1) \qquad P_i = P_{n-i}$$

and the generalized Euler theorem

$$(2) \qquad \sum_0^n (-1)^i \alpha_i = 1 + (-1)^n + \sum_1^{n-1} (-1)^i (P_i - 1),$$

where the α's have the meaning defined in § 4 below. The numbers P_i associated with a one-sided manifold satisfy the relation

* Read before American Mathematical Society, February 22, 1913.

† Annali di Matematica (2), vol. 4 (1871), p. 140.

‡ Werke, 2d edition, p. 474.

§ Journal de l'Ecole Polytechnique, vol. 1 (1895), p. 1; Rendiconti del Circolo Matematico di Palermo, vol. 13 (1899), p. 285; Proceedings of London Math. Society, vol. 32 (1900), p. 277.

|| A definition of these constants is given in § 14 below together with a proof of formulas (2) and (3).

$$\sum_{0}^{n}(-1)^i\alpha_i = 1 + \sum_{1}^{n-1}(-1)^i(P_i - 1) \tag{3}$$

and do not satisfy the duality theorem.*

2. In this paper, we attempt to establish some of the fundamental definitions and theorems as simply and rigorously as possible, so as to furnish an introduction to the memoirs of Poincaré. At the same time, we propose to show that a manifold may be described by means of certain systems of linear equations reduced modulo 2.* These equations lead us to a set of space constants

$$R_1, \quad R_2, \quad R_3, \quad \cdots, \quad R_{n-1}$$

which closely resemble the numbers P of Poincaré, both with respect to the geometric interpretation which may be given to them and in that they satisfy a duality theorem

$$R_i = R_{n-i} \tag{4}$$

and a generalized Euler theorem

$$\sum_{0}^{n}(-1)^i\alpha_i = 1 + (-1)^n + \sum_{1}^{n-1}(-1)^i(R_i - 1). \tag{5}$$

These relations hold good whether the manifold is one- or two-sided.

The numbers R_i are connected with the numbers P_i by a formula (§ 15) which involves the coefficients of torsion. They therefore do not supply us with any new invariants of a manifold. They seem to us, however, to deserve attention because they are connected in a fundamental way with the definition of a manifold and because of the generality and simplicity of the relations (3) and (4).

In §§ 17, 18, we show how a manifold may be described in a simple manner by means of a single matrix.

The Cell and the Complex.

3. An *n-dimensional simplex* will be defined as that one among the regions into which n-space is subdivided by $n + 1$ linearly independent $(n - 1)$-spaces which does not contain a point at infinity. Thus, the interior of a triangle in a plane is a two-dimensional simplex, and the linear segment joining two points is a one-dimensional simplex.

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* These equations are generalizations of those employed by O. Veblen in pp. 86–94 of this volume of the *Annals*. The connection of the operation of reducing modulo 2 with the definition of a manifold seems first to have been noted by H. Tietze, Monatshefte für Mathematik und Physik, vol. 19 (1908), p. 49. Tietze defines a set of constants Q_i analogous to the R's but which do not satisfy either a duality or an Euler theorem. They are intended for a different purpose from ours.

Now let $[P]$ be a set of objects (e. g., points or lines,—we shall always refer to these objects as points) such that there exists a one-to-one reciprocal correspondence between $[P]$ and the points interior to and on the boundary of an n-dimensional simplex. Then the points of $[P]$ which correspond to the interior of the simplex are said to constitute an *n-dimensional cell* E_n, and those which correspond to the boundary of the simplex are said to constitute the *boundary of the cell.*

We define the order relations among the points of E_n and its boundary in terms of the order relations among the images of these points on the simplex and its boundary. Moreover, when we say that a cell a is on the boundary of another cell b, we always mean to imply that the correspondence between the points of a and those of the simplex defining a is continuous with respect to the order relations among the points of b and its boundary.

4. Now consider a set C_n of cells consisting of

α_0	0-cells (points)	x_1^0,	x_2^0,	x_3^0,	$\cdots$,	$x_{\alpha_0}^0$
α_1	1-cells (arcs of curve)	x_1^1,	x_2^1,	x_3^1,	$\cdots$,	$x_{\alpha_1}^1$
α_2	2-cells	x_1^2,	x_2^2,	x_3^2,	$\cdots$,	$x_{\alpha_2}^2$
. . .	. . . . . .	.	.	.	.	.
α_i	i-cells	x_1^i,	x_2^i,	x_3^i,	$\cdots$,	$x_{\alpha_i}^i$
. . .	. . . . . .	.	.	.	.	.
α_n	n-cells	x_1^n,	x_2^n,	x_3^n,	$\cdots$,	$x_{\alpha_n}^n$

The set C_n will be called a *complex* provided the following conditions are satisfied:

(1) The boundary of every i-cell ($i > 0$) is made up entirely of cells x_k^j of dimensionalities less than i.

(2) Every i-cell ($i < n$) is on the boundary of some ($i + 1$)-cell x_k^{i+1}.

For example, the figure which is obtained when a projective 3-space is divided up into cells by means of a tetrahedron is a complex of a simple type. It is made up of four points, twelve 1-cells, sixteen 2-cells, and eight 3-cells. Thus, $\alpha_0 = 4$, $\alpha_1 = 12$, $\alpha_2 = 16$, and $\alpha_3 = 8$.

Description of a Complex By Means of Matrices.

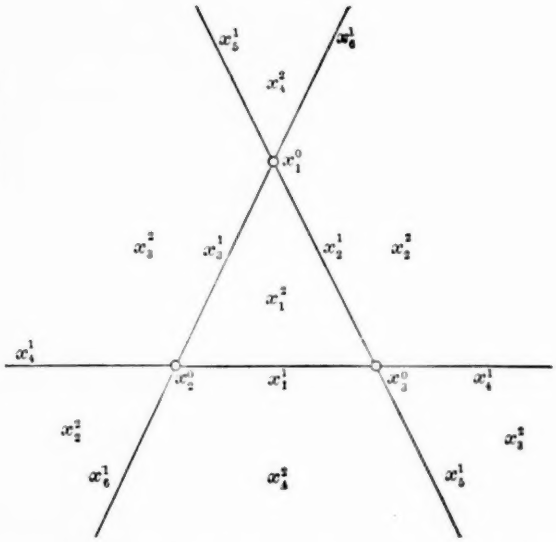
5. The structure of any n-dimensional complex C_n may be completely described by means of n suitably chosen tables or matrices $X_{0,1}$, $X_{1,2}$, $X_{2,3}$, $\cdots$, $X_{n-1,n}$. The matrix $X_{i-1,i}$ is an array of α_i columns and α_{i-1} rows, each column being associated with a distinct i-cell, and each row being associated with a distinct ($i - 1$)-cell of the complex C_n. In the jth row and kth column, there appears a number η_{jk}^i which is equal to

unity if the cells which correspond to the jth row and kth column respectively are adjacent, but otherwise equal to zero.*

6. On page 87 of this volume of the *Annals* are given the matrices which describe the surface of a tetrahedron. We give below the matrices for the real projective plane when subdivided into triangles by means of three straight lines.

	x_1^1	x_2^1	x_3^1	x_4^1	x_5^1	x_6^1
x_1^0	0	1	1	0	1	1
x_2^0	1	0	1	1	0	1
x_3^0	1	1	0	1	1	0

	x_1^2	x_2^2	x_3^2	x_4^2
x_1^1	1	0	0	1
x_2^1	1	1	0	0
x_3^1	1	0	1	0
x_4^1	0	1	1	0
x_5^1	0	0	1	1
x_6^1	0	1	0	1



To describe the complex which we mentioned at the end of § 4 will require three matrices. The first will have four rows and twelve columns; the second, twelve rows and sixteen columns; and the third, sixteen rows and eight columns.

7. A complex C_n is said to be *closed* if every one of its $(n-1)$-cells is upon the boundary of an even number of n-cells; otherwise, it is said to be *open* or *bounded*, and its boundary is said to consist of those $(n-1)$-cells (together with their boundaries) upon which an odd number of n-cells abut.

If a closed complex C_n contains no closed n-dimensional sub-complex, it is called an *n-dimensional circuit*. A 1-circuit, according to this definition, is a simple curve composed of a chain of arcs.

8. Let us remark in passing that Poincaré sometimes permits the boundary of an i-cell to touch itself along one or more cells of lower dimensionalities. When a complex contains cells of this more general type, it cannot always be completely characterized by means of the matrices $X_{0,1}, X_{1,2}, X_{2,3}, \cdots, X_{n-1,n}$.† The description may be made somewhat

* The relation between these matrices and the matrices of Poincaré (Lond. Math. Soc. Proc., vol. 32, p. 280) is that $\eta^i_{jk} = |\epsilon^i_{kj}|$. Clearly, the sign of η^i_{jk} is not relevant to the *definition* of a manifold.

† The same may be said about the matrices of Poincaré.

more precise if, in constructing the matrices, we make use of integers other than 1 whenever we wish to indicate that an i-cell appears more than once upon the boundary of an $(i + 1)$-cell, but a simple example will show that ambiguity may arise even in this case.

Take two triangles ABC and $A'B'C'$ and deform them in such a way that the six vertices $ABCA'B'C'$ coincide. A complex C_2 may then be obtained if the arcs AB, BC, and CA be deformed into coincidence with the arcs $A'B'$, $B'C'$, and $C'A'$ respectively. If however we deform the arcs AB, BC, and CA into coincidence with $B'C'$, $A'B'$, and $C'A'$ respectively, we obtain a different complex $\bar{C}_2$ which determines the same matrices as C_2. We may verify that C_2 and $\bar{C}_2$ are distinct by observing that the neighborhood of the vertex of C_2 has three sheets, whereas the neighborhood of the vertex of $\bar{C}_2$ has two.

Manifolds.

9. The totality of points in the various cells of a complex C_n constitutes an ordered set $[P]$. This set will be called a *manifold* M_n if and only if it has the following three properties.

1. That every point P is interior to some n-cell of $[P]$.* (This condition is obviously satisfied by all the points interior to the n-cells of C_n, but would not in general be satisfied by the points on their boundaries.)

2. That if two n-cells E_n^1 and E_n^2 of $[P]$ have a point in common, there exists an n-cell contained within each of the cells E_n^1 and E_n^2.

3. That if P and P' be any two points of C_n, there always exists a chain of overlapping n-cells connecting an n-cell about P to an n-cell about P'.

In reading these conditions, it should be remembered that a cell, by definition, contains no point of its own boundary. The conditions imply that every $(n - 1)$-cell x_i^{n-1} of a manifold is on two and only two n-cells, that every x_i^{n-2} is on a set of x_j^n's and x_k^{n-1}'s which are related like a set of points and arcs making up a circle, that every x_i^{n-3} is on a set of x_j^n's, x_k^{n-1}'s, and x_l^{n-2}'s which are related like the points, arcs, and spherical regions forming the surface of a sphere, and so on. Among the possibilities which are excluded are surfaces such as the complete cone, because of the singularity at the apex, and complexes such as the one obtained by partitioning the surface of an anchor ring into cells, selecting a point P not on the 3-space of the anchor ring and erecting upon each cell of the anchor ring as base a pyramid with P as apex.†

* When we say that a is a cell of $[P]$, we always imply that the correspondence between a and the simplex which defines a is continuous with respect to the order relations among the points of $[P]$ (cf. § 3).

† Encyklopädie der Mathematischen Wissenschaften, III, A, B, 3 (Dehn and Heegard), p. 183.

10. It is evident that many complexes give rise to the same or equivalent manifolds, for every manifold may be partitioned into cells in an indefinitely large number of ways. Hence the problem arises to determine certain invariants which characterize the matrices of all complexes associated with the same manifold.

The problem has been completely solved for the case $n = 2$, for it has been found that a manifold M_2 is completely determined when we know whether it is one or two-sided* and what is the value of the expression $\alpha_0 - \alpha_1 + \alpha_2$. For the cases where n is greater than 2, however, no complete set of invariants has yet been discovered.

The Generalized Euler Formula and the Constants *R*.

11. Let us regard the symbols x^i_i associated with the various rows and columns of the matrices as variables which may take on the values 0 and 1, and which are combined by reducing modulo 2. Then corresponding to every jth row of the matrix $X_{i-1,\,i}$ we shall write an equation of the form

$$(X_{i-1,\,i}) \qquad \sum_{k=1}^{\alpha_i} \eta^i_{jk} x^i_k = 0 \qquad j = 1, 2, 3, \cdots \alpha_{i-1},\dagger$$

where we recall that η^i_{jk} is the number which appears in the jth row and kth column of the matrix $X_{i-1,\,i}$, and where $x^i_1, x^i_2, x^i_3, \cdots, x^i_{\alpha_i}$ are the variables which correspond to the various columns of $X_{i-1,\,i}$. In this manner, a modular equation is associated with every $(i-1)$-cell of C_n, the right-hand member of which is zero and the left-hand member of which consists of the sum of the symbols x^i_k corresponding to the i-cells which abut upon the $(i-1)$-cell in question. (The coefficients of the remaining i-cells are all zero.)

Now, a solution of the system of equations $(X_{i-1,\,i})$ marks every i-cell of C_n with a number 0 or 1. Moreover, since no equation of the set is satisfied unless an even number or none of its variables take on the value 1, we see that among the i-cells which abut upon any given $(i-1)$-cell of the complex, an even number or none will always be marked with a 1. In other words, every solution of $(X_{i-1,\,i})$ defines an i-circuit or a system of i-circuits; and, conversely, every i-circuit or system of i-circuits defines a solution of $(X_{i-1,\,i})$. In particular, every column of the matrix $X_{i,\,i+1}$ defines a solution of the system $(X_{i-1,\,i})$, for it marks with a 1 all of the i-cells upon the boundary of one of the $(i+1)$-cells. The converse is obviously not true, since there are in general many circuits which are not the boundaries of cells of C_n.

*The distinction between one- and two-sidedness will be made in § 13.

† These equations have been explained in a simple case on pp. 86–94 of this volume of the *Annals*.

If two solutions s_1 and s_2 be added, their sum is a solution s_3 which is linearly dependent upon s_1 and s_2. Geometrically, s_3 will be represented by the set of circuits which one obtains by superimposing the circuits s_1 upon the circuits s_2 and leaving off the i-cells which are common to the two systems. Thus, if the boundaries of a set of $(i+1)$-cells be added according to the above rule, the resulting system of i-circuits will be the boundary of one or more $(i+1)$-complexes (§ 7) consisting of the $(i+1)$-cells in question. Conversely, the boundaries of one or more open $(i+1)$-complexes can always be expressed as a sum of the boundaries of the $(i+1)$-cells which make up the $(i+1)$-complexes.

12. We shall denote the maximum number of linearly independent solutions of $(X_{i-1,\,i})$ by $\sigma_{i-1,\,i}$. Then the total number of solutions will be $2^{\sigma_{i-1,\,i}}$, since every solution of $(X_{i-1,\,i})$ is of the form

$$\sum_{j=1}^{\sigma_{i-1,\,i}} \lambda_j c_j \qquad (\lambda = 0, 1)$$

where $c_1, c_2, c_3, \cdots, c_{\sigma_{i-1,\,i}}$ are a complete set of linearly independent solutions.

Now by a well-known theorem which is true for modular as well as for ordinary linear equations,

$$\sigma_{i-1,\,i} = \alpha_i - \rho_{i-1,\,i}$$

where $\rho_{i-1,\,i}$ is the order of the non-vanishing determinant of highest order in the matrix $X_{i-1,\,i}$ (i. e., the rank of $X_{i-1,\,i}$), and α_i is defined as in § 4. The count may also be made in another way. For we have seen that every column of the matrix $X_{i,\,i+1}$ yields a solution of the equations $(X_{i-1,\,i})$, and since the rank of $X_{i,\,i+1}$ is $\rho_{i,\,i+1}$, the number of linearly independent solutions of this type must also be $\rho_{i,\,i+1}$. Combinations of these solutions give the boundaries of open $(i+1)$-complexes, as we have seen. In general, there will also exist non-bounding i-circuits which cannot be expressed linearly in terms of the boundaries of cells. Let $(R_i - 1)$ *be the number of i-circuits which must be added to the bounding i-circuits before we can obtain a complete set of linearly independent solutions.* Then

$$\sigma_{i-1,\,i} = \rho_{i,\,i+1} + (R_i - 1),$$

and, equating the two values for $\sigma_{i-1,\,i}$,

$$\alpha_i - \rho_{i-1,\,i} = \rho_{i,\,i+1} + (R_i - 1).$$

The value of $\rho_{0,\,1}$ may be determined directly by special considerations. For the number 1 will appear just twice in every column of the matrix $X_{0,\,1}$ since every arc has two end points. Consequently, if we add together all the equations $(X_{0,\,1})$, their sum must be identically zero, modulo 2;

or in other words, the last column must be equal to the sum of the first $\alpha_0 - 1$ columns, and $\rho_{0,1}$ cannot be greater than $\alpha_0 - 1$. Nor is $\rho_{0,1}$ less than $\alpha_0 - 1$. For if it were, the sum of some subset $(X'_{0,1})$ of the equations would vanish identically, and we could subdivide the vertices of C_n into two classes, the first being made up of the vertices corresponding to the equations $(X'_{0,1})$ and the second, of the remaining ones. And there would be no arc x^1_i joining a vertex of the first class to one of the second class, otherwise the symbol x^1_i would appear once and only once in the equations $(X'_{0,1})$, and the sum of these equations could not vanish identically. Thus, if $\rho_{0,1}$ were less than $\alpha_0 - 1$, the manifold would not be connected, contrary to definition.

Thus, we have that

$$\alpha_0 - 1 = \rho_{0,1}.$$

Arguing in a similar way, we show that the rank $\rho_{n-1,n}$ of the matrix $X_{n-1,n}$ is $\alpha_n - 1$. For in every row, there are two 1's since two and only two n-cells abut upon the same $(n-1)$-cell. Consequently, the sum of all the columns vanishes identically, proving that $\rho_{n-1,n}$ is not greater than $\alpha_n - 1$. But the sum of a smaller number of columns than $\alpha_n - 1$ cannot vanish identically, otherwise these columns would define a set of n-cells having no $(n-1)$-cell in common with the remaining ones, which would again mean that C_n was not connected.

Hence,

$$\alpha_n - \rho_{n-1,n} = 1.$$

In all, we have the following relations:

$$\begin{aligned}
\alpha_0 - 1 &= \rho_{0,1},\\
\alpha_1 - \rho_{0,1} &= \rho_{1,2} + (R_1 - 1),\\
\alpha_2 - \rho_{1,2} &= \rho_{2,3} + (R_2 - 1),\\
&\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\\
\alpha_{n-2} - \rho_{n-3,n-2} &= \rho_{n-2,n-3} + (R_{n-2} - 1),\\
\alpha_{n-1} - \rho_{n-2,n-1} &= \rho_{n-1,n} + (R_{n-1} - 1),\\
\alpha_n - \rho_{n-1,n} &= 1.
\end{aligned} \tag{6}$$

Multiplying these equations alternately by $+1$ and -1 and adding, the ρ's disappear and we obtain the relation

$$\sum_0^n (-1)^i \alpha_i = 1 + (-1)^n + \sum_1^{n-1} (-1)^i (R_i - 1), \tag{5}$$

which is the same as (5), § 1.

The Notion of Sense.

13. Let us make the convention that every arc of the complex C_n shall be positively related to one of its end-points and negatively related to the other. We can then assign to every arc one of two "senses" according as we say that it is positively related to one or the other of its end-points.* The sensed arc x_i^1 will be denoted by the symbol $\pm a_i^1$, the positive sign being associated with one determination of sense, the negative sign with the other.

The way in which we choose to assign the senses to the various arcs of C_n may be indicated by modifying the matrix $X_{0,1}$. Leaving the symbol 0 unchanged wherever it occurs, we shall replace the symbol 1 by the symbol -1 whenever the arc and point to which it corresponds are negatively related. If the arc and point are positively related, we shall leave the 1 unchanged.

In every column of the new matrix $A_{0,1}$ which we thus obtain, the numbers 1 and -1 each appear once. The sign of one of the 1's in each of the columns may be selected arbitrarily, after which, the sign of the other will be determined. Every choice corresponds to a different way of giving senses to the arcs.

We shall say that two arcs which abut upon the same point P have the same sense if one is positively and the other negatively related to P, and that they have opposite senses if they are similarly related to P. Then it is easy to see that from any one-dimensional circuit, there may be derived two sensed circuits, where in a sensed circuit, every arc has the same sense as each of the two arcs upon which it abuts. For as soon as we assign a sense to one of the arcs, the senses of all the other arcs are uniquely determined. Either of two senses may thus be assigned to the boundary of a two-cell.

To assign sense to a 2-cell x_i^2, we may make the convention that x_i^2 is positively related to its boundary c taken in one sense and negatively related to its boundary $-c$ taken in the opposite sense. x_i^2 will then be said to be positively related to the cells of c and negatively related to the cells of $-c$. To give the opposite sense to x_i^2, we say that it shall be positively related to $-c$.

If we replace 1 by -1 in the matrix $X_{1,2}$ whenever we wish to indicate that a 2-cell and an adjacent 1-cell are negatively related, we shall obtain a matrix $A_{1,2}$ in which the senses of the 2-cell are indicated. If the sense of any 1-cell be changed, so must be the signs of all the 1's in the corresponding row of $A_{1,2}$, while if the sign of a 2-cell be changed, so must be the signs of the 1's in the corresponding column.

* The connection of this statement with the intuitive notion of sense is obvious.

Now if we make the convention that two adjacent 2-cells have the same or opposite senses according as they are oppositely or similarly related to a common arc, we cannot conclude as we could in the linear case that two sensed 2-circuits can always be derived from an unsensed circuit c_2. When we can, c_2 is said to be *two-sided*; when we cannot, c_2 is said to be *one-sided*. The boundary of a 3-cell is two-sided; the projective plane, one-sided, as may be verified by an examination of the matrices in § 6.

Finally, we may assign senses to the 3-cells, 4-cells, $\cdots$, n-cells of C_n just as we did to the 2-cells, and can obtain a set of matrices $A_{0,1}$, $A_{1,2}$, $A_{2,3}$, $A_{3,4}$, $\cdots$, $A_{n-1,n}$ which not only define the complex C_n but also the sense of every cell of C_n (except the 0-cells or points, which are not assigned senses).

14. Now, let us consider a set of ordinary *non-modular* linear equations

$$(A_{i-1,\,i}) \qquad \sum_{k=1}^{\alpha_i} \epsilon_{jk}{}^i x_k{}^i = 0 \qquad j = 1, 2, 3, \cdots, \alpha_{i-1},$$

which can be derived from the matrix $A_{i-1,\,i}$ just as the equations $(X_{i-1,\,i})$ were derived from the matrix $X_{i-1,\,i}$.

Then every solution of the system $(A_{i-1,\,i})$ in integers will correspond to a sensed i-circuit or system of i-circuits, provided that we make the convention that in a circuit the same cell may appear and be counted more than once. Moreover, every solution of $(A_{i-1,\,i})$ is linearly dependent upon a set of integer solutions, and hence we may say that the number of linearly independent sensed (and therefore two-sided) circuits is equal to the number of linearly independent solutions of $(A_{i-1,\,i})$.

The operation of combining sensed circuits which corresponds to adding solutions of the above equations is not the same as the operation of combining unsensed ones which we previously considered. For if two sensed circuits having a common cell are added, the coincident cells do not annul one another unless they have opposite senses.

Now, by the theorem on linear equations which we used when working with the modular equations, if $\mu_{i-1,\,i}$ be the number of linearly independent solutions of the system $(A_{i-1,\,i})$, then

$$\mu_{i-1,\,i} = \alpha_i - \nu_{i-1,\,i}$$

where $\nu_{i-1,\,i}$ is the rank of the matrix $A_{i-1,\,i}$. Furthermore, since the boundary of every $(i+1)$-cell taken in a definite sense is an i-circuit, the columns of the matrix $A_{i,\,i+1}$ must define solutions of the system $(A_{i-1,\,i})$. And since the rank of $A_{i,\,i+1}$ is $\nu_{i,\,i+1}$, the number of linearly independent solutions of this sort is $\nu_{i,\,i+1}$. *Let $(P_i - 1)$ be the number of sensed i-circuits*

which must be added to the bounding i-circuits before we can obtain a complete set of linearly independent solutions. Then the relation

$$\mu_{i-1,\,i} = \nu_{i,\,i+1} + (P_i - 1)$$

must also hold. Equating the two values of $\mu_{i-1,\,i}$ we have

$$\alpha_i - \nu_{i-1,\,i} = \nu_{i,\,i+1} + (P_i - 1).$$

The values of $\nu_{0,\,1}$ and $\nu_{n-1,\,n}$ may be calculated directly by special considerations analogous to those used in determining $\rho_{0,\,1}$ and $\rho_{n-1,\,n}$ in connection with the modular equations. The sum of all the equations $(A_{0,\,1})$ vanishes identically, whereas if a linear combination of less than all vanished, the vertices could be divided into sets corresponding to disconnected portions of C_n. Hence,

$$\alpha_0 - 1 = \nu_{0,\,1}.$$

Furthermore, no linear combination of less than all the columns of the last matrix $A_{n-1,\,n}$ can vanish; otherwise, the n-cells defined by such columns would not be connected with the rest of the manifold by any $(n-1)$-cell. We cannot conclude, however, that there exists a linear combination of all of the columns which vanishes identically. Two cases arise according as C_n is one- or two-sided. If C_n is two-sided, then senses may be so assigned to the various n-cells that every $(n-1)$-cell is oppositely related to the two n-cells which abut upon it. In other words, the sum of the columns, with the proper sign given to each, vanishes, and we have

$$\nu_{n-1,\,n} = \alpha_n - 1.$$

For one-sided complexes no such relation exists and

$$\nu_{n-1,\,n} = \alpha_n.$$

Thus, we have a criterion for one- and two-sidedness.

Altogether, we have the relations

$$\begin{aligned}
\alpha_0 - 1 &= \nu_{0,\,1}, \\
\alpha_1 - \nu_{0,\,1} &= \nu_{1,\,2} + (P_1 - 1), \\
\alpha_2 - \nu_{1,\,2} &= \nu_{2,\,3} + (P_2 - 1), \\
&\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots \\
\alpha_{n-1} - \nu_{n-2,\,n-1} &= \nu_{n-1,\,n} + (P_{n-1} - 1), \\
\left.\begin{aligned}\alpha_n - \nu_{n-1,\,n} &= 1 \\ \text{or}\quad \alpha_n - \nu_{n-1,\,n} &= 0\end{aligned}\right\}&.
\end{aligned} \tag{7}$$

From which we derive the formulas

(2) $$\sum_0^n (-1)^i \alpha_i = 1 + (-1)^n + \sum_1^{n-1} (-1)^i (P_i - 1)$$

and

(3) $$\sum_0^n (-1)^i \alpha_i = 1 + \sum_1^{n-1} (-1)^i (P_i - 1)$$

for two- and one-sided manifolds respectively. These are the Euler-Poincaré equations, and the numbers

$$P_1, \quad P_2, \quad P_3, \quad \cdots, \quad P_{n-1}$$

are the Betti numbers of Poincaré.*

Relation between the Numbers R and P.

15. Let $X_{0,1}, X_{1,2}, \cdots, X_{n-1,n}$ be a set of matrices of the first type (§ 5) which describe a manifold M_n, and let $A_{0,1}, A_{1,2}, \cdots, A_{n-1,n}$ be the corresponding set of the second type (§ 13). Then each of the matrices $A_{i,i+1}$ may be reduced by means of elementary transformations "without division"† to a matrix $\bar{A}_{i,i+1}$ where all the elements of $\bar{A}_{i,i+1}$ which are not on the main diagonal are zero and the elements of the main diagonal are the invariant factors of $A_{i,i+1}$. The rank of $A_{i,i+1}$ is the same as the number of non-vanishing elements of $\bar{A}_{i,i+1}$. Whenever a number other than 0 or 1 appears in one of the reduced matrices, the absolute value of that number is said to be a *coefficient of torsion* of the manifold M_n. The coefficients of torsion are invariants of M_n.‡

Now the rank of the matrix $X_{i,i+1}$ *modulo 2* is the same as that of the matrix $A_{i,i+1}$ *modulo 2*, for the elements of $X_{i,i+1}$ differ at most in sign from the corresponding elements of $A_{i,i+1}$. Hence, the rank of $X_{i,i+1}$ is equal to the rank of $\bar{A}_{i,i+1}$ after the elements along the diagonal of $\bar{A}_{i,i+1}$ have been reduced modulo 2. But the effect of this reduction is to replace the even coefficients of torsion by 0's and the odd ones by 1's. Hence,

(8) $$\nu_{i,i+1} - \rho_{i,i+1} = t_{i,i+1}$$

where $\nu_{i,i+1}$ is the rank of $A_{i,i+1}$; $\rho_{i,i+1}$, the rank of $X_{i,i+1}$ modulo 2; and $t_{i,i+1}$ the number of even coefficients of torsion of $A_{i,i+1}$.

But in §§ 12 and 14 respectively, we derived the two relations

(6) $$\alpha_i - \rho_{i-1,i} = \rho_{i,i+1} + (R_i - 1),$$

and

(7) $$\alpha_i - \nu_{i-1,i} = \nu_{i,i+1} + (P_i - 1),$$

whence we have

* Palermo Rendiconti, vol. 13 (1899), p. 286 and p. 301.

† Poincaré, Lond. Math. Soc. Proc., vol. 32 (1900), p. 286.

‡ Loc. cit., p. 286 et seq.

$$R_i - P_i = (\nu_{i-1,\,i} - \rho_{i-1,\,i}) + (\nu_{i,\,i+1} - \rho_{i,\,i+1})$$

or

$$R_i = P_i + t_{i-1,\,i} + t_{i,\,i+1}, \tag{9}$$

which gives the relation between the constants R and the constants P found by Poincaré. In comparing equations (2) and (3) with (5) by means of (9), it must be remembered that $t_{0,\,1} = 0$ and $t_{n-1,\,n}$ is 0 or 1 according as the manifold is two- or one-sided.

16. Since Poincaré has shown that the Betti numbers and the coefficients of torsion are invariants of a manifold,* the invariance of the numbers R follows at once from Equation (9). And since he has also shown that for two-sided manifolds

$$P_i = P_{n-i}$$

and

$$t_{i-1,\,i} = t_{n-i,\,n-i+1},$$

it also follows that for two-sided manifolds,

$$R_i = R_{n-i}.$$

That this last relation also holds for one-sided manifolds will follow as a result of the discussion in § 18.

Regular Subdivision.

17. We have seen that every manifold M_n may be described by means of the n matrices associated with any complex C_n which defines M_n. We shall now prove that if the cells of C_n be suitably subdivided, a complex $\bar{C}_n$ may always be obtained which is of such a simple type that it is completely characterized by a single matrix. A subdivision of the required sort is given below:

First, we introduce a new vertex upon each arc of C_n, thereby subdividing it into two arcs, *no two of which are bounded by the same pair of end-points.* For convenience, we shall call these new arcs one-dimensional " pyramids." The apexes of these " pyramids " will be the new vertices and the bases will be the old vertices of C_n.

Secondly, we introduce a new vertex upon every 2-cell of C_n and join each of these vertices by arcs to the vertices upon the boundary of the cell in which it lies. The 2-cells are thus decomposed into triangular regions or two-dimensional " pyramids " which are bounded by arcs or one-dimensional pyramids. The new vertices are the apexes of the new pyramids, the old points and arcs, their bases.

* It must be remarked here that Poincaré assumes in proving the invariance of the P's that every cell of C_n is made up of a finite number of analytic pieces.

Thirdly, by a similar process, we subdivide the 3-cells into three-dimensional pyramids bounded by one- and two-dimensional ones, and so on.*

After we have subdivided all of the cells of C_n in the above manner, we shall have a complex $\bar{C}_n$ of a very simple type. For, every two-cell of $\bar{C}_n$ is bounded by three arcs; every 3-cell, by four triangular faces; every 4-cell, by five tetrahedral faces; and so on. In other words, not only will it be possible to map every k-dimensional cell E_k of $\bar{C}_n$ along with its boundary upon the interior and boundary of a k-dimensional simplex S_k, but the mapping may be made in such a way that the image of every 0-cell on the boundary of E_k is a vertex of S_k; the image of every 1-cell on the boundary of E_k, an edge of S_k; the image of every 2-cell on the boundary of E_k, a face of S_k; and so on.

Let us also observe that if E_i^1 and E_i^2 be any two i-cells of $\bar{C}_n$, then the vertices of $\bar{C}_n$ which lie upon the boundary of E_i^1 cannot all lie upon the boundary of E_i^2. If E_i^1 and E_i^2 are arcs, the truth of this statement is obvious from the construction of $\bar{C}_n$; in the higher cases, the proof may be made as follows:

Suppose the same $i + 1$ vertices appeared upon the boundaries of both E_i^1 and E_i^2. Then that one of the $i + 1$ vertices which was the last to be introduced during the subdivision of C_n would necessarily be the apex of both the "pyramids" E_i^1 and E_i^2, while the remaining i vertices would lie upon the boundaries of the bases of E_i^1 and E_i^2. Calling these bases E_{i-1}^1 and E_{i-1}^2 respectively, we could apply the argument just made to these two new cells and thereby show that there existed two $(i - 2)$-cells E_{i-2}^1 and E_{i-2}^2 upon whose boundaries the same $i - 1$ vertices of $\bar{C}_n$ appeared. And after $i - 1$ steps of this sort, we should be led to the conclusion that two arcs E_1^1 and E_1^2 of $\bar{C}_n$ were bounded by the same pair of points.

18. Now suppose we consider a simplex S having the same number of vertices as the complex $\bar{C}_n$. Then if we denote the vertices of $\bar{C}_n$ by

$$x_1^0, \quad x_2^0, \quad x_3^0, \quad \cdots, \quad x_p^0$$

and the vertices of S by

$$V_1, \quad V_2, \quad V_3, \quad \cdots, \quad V_p,$$

it is clear that to every vertex x_i^0 of $\bar{C}_n$ may be made to correspond the vertex V_i of S. And in a like manner, to every one-cell $x_i^0 x_j^0$ of $\bar{C}_n$ may be made to correspond the one-cell $V_i V_j$ of S; to every 2-cell $x_i^0 x_j^0 x_k^0$, the 2-cell $V_i V_j V_k$; and so on. Moreover, no two cells of $\bar{C}_n$ are made to correspond to the same cell of S, otherwise the boundaries of both would include the same vertices of C_n, which we proved to be impossible.

* Cf. Poincaré, Palermo Rendiconti, vol. 13, p. 314.

We thus have the theorem that *every complex may be subdivided into a complex $\bar{C}_n$ which is equivalent to a complex D_n the elements of which are a subset of the elements upon the boundary of a simplex of $p - 1$ dimensions.*

Now a complex of the type $\bar{C}_n$ may be described by means of a single matrix X the rows and columns of which are in one-to-one correspondence with the vertices and n-cells of $\bar{C}_n$ respectively. We put 1 or 0 in the ith row and jth column according as the point x_i^0 is or is not on the n-cell x_j^n. To prove that the matrix X describes the complex $\bar{C}_n$ we need only observe that every k-cell of $\bar{C}_n$ is determined by $k + 1$ vertices which lie upon the boundary of the same n-cell of $\bar{C}_n$; while, conversely, every set of $k + 1$ vertices which lie upon the boundary of the same n-cell determines one and only one k-cell. Consequently, $k + 1$ given vertices of $\bar{C}_n$ will determine a k-cell if and only if there exists a column in the matrix X which contains a 1 in each of the rows corresponding to the vertices in question.

Dual Complexes.

19. If C_n be a complex which defines a manifold M_n, there always exists a complex C'_n which is dual to C_n and which also defines M_n; where two complexes C_n and C'_n are said to be *dual* if the points, arcs, $\cdots$, k-cells, $\cdots$ of the one are in one-to-one correspondence with the n-cells, $(n - 1)$-cells, $\cdots$, $(n - k)$-cells, $\cdots$ respectively of the other, the correspondence being such that two cells of C'_n abut upon one another if and only if the corresponding cells of C_n do.

A simple way of showing the existence* of C'_n is to observe that if a regular subdivision be applied to the complex C_n, a complex $\bar{C}_n$ is thereby obtained which contains the required complex C'_n as a sub-complex. C'_n may in fact be derived from $\bar{C}_n$ by applying what we may term the inverse of a regular subdivision.

For, by the definition of a manifold, the n-cells of $\bar{C}_n$ which abut upon one of the vertices V_0 of the original complex C_n constitute, with the cells which separate them from one another, an n-cell E_n. The inverse of a regular subdivision applied to these cells will amount to replacing the vertex V_0 and the adjacent cells by the single cell E_n. By making a similar transformation about every vertex of $\bar{C}_n$ which also belongs to the original complex C_n, we shall obtain a set of n-cells which are in one-to-one correspondence with the vertices of C_n and which will serve as the n-cells of the dual complex C'_n.

Upon the boundaries of the n-cells E_n will appear the points V_1 which we introduced in subdividing the arcs of C_n. We can apply the inverse of a regular subdivision to the remaining cells of dimensionality $n - 1$ or less which

* Poincaré, Palermo Rendiconti, vol. 13, p. 314.

cluster about each of these points V_1, thereby obtaining new $(n-1)$-cells which are in one-to-one correspondence with the arcs of C_n and will serve as the $(n-1)$-cells of C'_n. After the $(i+1)$st step, there will remain upon the boundaries of the $(n-i)$-cells E_{n-i} the points V_{i+1} which we introduced upon the $(i+1)$-cells of C_n when we passed from C_n to $\bar{C}_n$. The inverse of a regular subdivision applied to the cells of dimensionality $n-i-1$ and lower which cluster about the vertices V_{i+1} will give the cells E_{n-i-1} of C'_n. Thus we may construct a complex C'_n dual to C_n.

Let us denote by $X_{0,1}$, $X_{1,2}$, $X_{2,3}$, $\cdots$, $X_{n-1,n}$ the matrices of the complex C_n, and by $X'_{0,1}$, $X'_{1,2}$, $X'_{2,3}$, $\cdots$, $X'_{n-1,n}$ the matrices of C'_n. Then owing to the fact that there is a one-to-one correspondence between the i-cells of C_n and the $(n-i)$-cells of C'_n and that an i-cell of C_n abuts upon an $(i+1)$-cell if and only if the corresponding $(n-i)$-cell of C'_n abuts upon the corresponding $(n-i-1)$-cell, it follows that the matrix $X'_{n-i-1,\,n-i}$ is the same as the matrix $X_{i,\,i+1}$ with rows and columns interchanged, and hence that the two matrices have the same rank.

Now, from Equations (6), § 12, we have at once

$$\begin{aligned} R_i &= \alpha_i - \rho_{i-1,\,i} - \rho_{i,\,i+1}, \\ &= \alpha'_{n-i} - \rho'_{n-i,\,n-i+1} - \rho'_{n-i-1,\,n-i} = R'_{n-i}, \end{aligned}$$

where R_i and R'_i denote the space constants of C_n and C'_n respectively.

But owing to the invariance of the constants of a manifold, we have

$$R_{n-i} = R'_{n-i},$$

and hence

$$R_i = R_{n-i},$$

which proves the duality relation for both one- and two-sided manifolds.

The proof which we have given above is essentially the one which Poincaré uses in showing that

$$P_i = P_{n-i}$$

for two-sided manifolds. Having proved the invariance of the coefficients of torsion, it may also be used in proving the duality relation which the latter also satisfy for two-sided manifolds. Similar theorems do not hold for one-sided manifolds; for although a dual complex C'_n may always be found, as was shown above, we cannot in the one-sided case so assign the senses to the cells of C_n and C'_n that corresponding cells are similarly sensed.

PRINCETON, 1913.

SYSTEMS OF PLANE CURVES WHOSE INTRINSIC EQUATIONS ARE ANALOGOUS TO THE INTRINSIC EQUATION OF AN ISOTHERMAL SYSTEM.*

By H. W. Reddick.

If we take the differential equation of a singly infinite system of plane curves in the form

$$y' = \tan \lambda(x, y), \tag{1}$$

the curvature κ of a curve of the system and the curvature κ_1 of a curve of the orthogonal system are given by the formulas:

$$\kappa = \lambda_x \cos \lambda + \lambda_y \sin \lambda, \quad \kappa_1 = \lambda_y \cos \lambda - \lambda_x \sin \lambda, \tag{2}$$

where the subscripts x and y denote partial differentiation.

Connected with this system of curves and its orthogonal system are the four intrinsic quantities T, N, T_1 and N_1, defined as follows:

$T = \dfrac{d\kappa}{ds}$ = the rate of variation of the curvature of a curve of the given system (1) along the curve itself in the direction taken arbitrarily as positive.

$N = \dfrac{d\kappa}{dn}$ = the rate of variation of the curvature of a curve of the given system along a curve of the orthogonal system in the positive direction, which is taken to be the direction making an angle of $+ 90°$ with the positive direction of the curve of the given system.

$T_1 = \dfrac{d\kappa_1}{ds_1}$ = the rate of variation of the curvature of a curve of the orthogonal system along the curve itself in the positive direction.

$N_1 = \dfrac{d\kappa_1}{dn_1}$ = the rate of variation of the curvature of a curve of the orthogonal system along a curve of the given system in the direction making an angle of $+ 90°$ with the positive direction of the curve of the orthogonal system.

It is known that $T + T_1 = 0$† is the intrinsic equation of the important

* Presented at the meeting of the American Mathematical Society, October 26, 1912.

† See Kasner, "The Riccati Differential Equations which Represent Isothermal Systems," Bull. Am. Math. Soc., vol. 10, p. 342.

isothermal system. It would seem that equations analogous, from this point of view, to the intrinsic equation of an isothermal system (in particular, the equation $T - T_1 = 0$) might also represent interesting systems. It is the purpose of this paper to consider the twelve systems of curves whose intrinsic equations are formed by equating to zero the sums and differences of the four quantities T, N, T_1 and N_1, taken in pairs.

The Equations of the Twelve Systems.

We now enumerate the twelve systems as given by their intrinsic equations:

$$T + T_1 = 0, \quad (\mathrm{I}) \qquad N + T_1 = 0, \quad (\mathrm{VII})$$
$$N + N_1 = 0, \quad (\mathrm{II}) \qquad N_1 - T = 0, \quad (\mathrm{VIII})$$
$$T - T_1 = 0, \quad (\mathrm{III}) \qquad N_1 + T_1 = 0, \quad (\mathrm{IX})$$
$$N - N_1 = 0, \quad (\mathrm{IV}) \qquad N_1 - T_1 = 0, \quad (\mathrm{X})$$
$$N + T = 0, \quad (\mathrm{V}) \qquad N_1 + T = 0, \quad (\mathrm{XI})$$
$$N - T = 0, \quad (\mathrm{VI}) \qquad N - T_1 = 0. \quad (\mathrm{XII})$$

If we change T into T_1 and N into N_1, equations I, II, III and IV remain unaltered, which shows that each of these systems has the same intrinsic equation as its orthogonal system, but equations V, VI, VII and VIII are changed into IX, X, XI and XII respectively. Systems IX, X, XI and XII are therefore orthogonal to systems V, VI, VII and VIII respectively. It will be sufficient, then, to deal with the first eight systems.

The values of T, N, T_1 and N_1 in terms of λ and its partial derivatives are:

$$\begin{aligned} T &= [\alpha - (\gamma - \delta) \tan \lambda + \beta \tan^2 \lambda] \cos^2 \lambda, \\ N &= [\delta - (\alpha - \beta) \tan \lambda + \gamma \tan^2 \lambda] \cos^2 \lambda, \\ T_1 &= [\beta + (\gamma - \delta) \tan \lambda + \alpha \tan^2 \lambda] \cos^2 \lambda, \\ N_1 &= [\gamma + (\alpha - \beta) \tan \lambda + \delta \tan^2 \lambda] \cos^2 \lambda, \end{aligned} \tag{3}$$

where

$$\alpha = \lambda_{xx} + \lambda_x\lambda_y, \qquad \gamma = \lambda_x^2 - \lambda_{xy},$$
$$\beta = \lambda_{yy} - \lambda_x\lambda_y, \qquad \delta = \lambda_y^2 + \lambda_{xy}.$$

The equations of systems I–VIII may then be written in the form:

$$\alpha + \beta = \lambda_{xx} + \lambda_{yy} = 0, \tag{I'}$$
$$\gamma + \delta = \lambda_x^2 + \lambda_y^2 = 0, \tag{II'}$$
$$\alpha - \beta - (\gamma - \delta) \tan 2\lambda = 0, \tag{III'}$$
$$\alpha - \beta + (\gamma - \delta) \cot 2\lambda = 0, \tag{IV'}$$

$$\alpha + \delta - (\alpha - \beta + \gamma - \delta) \tan \lambda + (\beta + \gamma) \tan^2 \lambda = 0, \qquad (\mathrm{V}')$$
$$\alpha - \delta + (\alpha - \beta - \gamma + \delta) \tan \lambda + (\beta - \gamma) \tan^2 \lambda = 0, \qquad (\mathrm{VI}')$$
$$\beta + \delta - (\alpha - \beta - \gamma + \delta) \tan \lambda + (\alpha + \gamma) \tan^2 \lambda = 0, \qquad (\mathrm{VII}')$$
$$\alpha - \gamma - (\alpha - \beta + \gamma - \delta) \tan \lambda + (\beta - \delta) \tan^2 \lambda = 0. \qquad (\mathrm{VIII}')$$

The equations of systems IX–XII could be written down from those of V–VIII by changing $\tan \lambda$ into $- \cot \lambda$.

System I is the familiar isothermal system. The equation $y' = \tan \lambda (x, y)$, where λ is a solution of Laplace's Equation I′, represents all isothermal families of plane curves.*

System II is composed of all families of parallel straight lines in the plane, since we have from equation II′, $\lambda_x = \lambda_y = 0$, hence $\lambda =$ const., and the equation of the system is $y' =$ const., the ∞^1 families of parallel straight lines in the plane.

System IV is the 45° isogonal system of III, since equation IV′ becomes identical with equation III′ if λ is replaced by $(\pi/4) + \lambda$.

It remains to consider the five systems III, V, VI, VII and VIII. The general solution will not be found in these cases, but in each case a family of ∞^3 curves belonging to the system will be found. This family of ∞^3 curves will be composed of ∞^2 subfamilies of ∞^1 curves each, such that all the curves of any subfamily may be obtained from any one of them by translation. That is, we shall find all systems of types III, V, VI, VII and VIII admitting a group of translations. We shall also find in the last section a characteristic geometric property of the complete system III.

We notice that $\lambda =$ const. satisfies the equation of each system, hence $y' =$ const. is a particular solution in each case. This solution will be disregarded in the following discussion.

We also note one obvious particular solution of equation III′, namely $T = T_1 = 0$. The equation $T = 0$ represents all circles in the plane (including straight lines) since a circle is the only real plane curve whose curvature is constant. The equation $T_1 = 0$ represents a system of curves whose orthogonal curves are circles. Hence $T = T_1 = 0$ represents all systems of circles whose orthogonal curves are also circles. Such systems are also isothermal, since $T = T_1 = 0$ is a solution of equation I′.

Systems of Type III Admitting a Group of Translations.

Systems admitting a group of translations will be of the form

$$y' = \tan \lambda(x)$$

* Lie-Scheffers, Differentialgleichungen, p. 157.

if the y-axis is taken as the direction of translation. To obtain such systems we make λ a function of x alone in equation III′. This gives

$$\lambda'' - \lambda'^2 \tan 2\lambda = 0, \tag{4}$$

the primes denoting derivatives with respect to x. The first integral of (4) is easily found to be

$$\lambda' \sqrt{\cos 2\lambda} = c_1. \tag{5}$$

The integration of equation (5) involves elliptic integrals. Writing it in the form

$$c_1 dx = \sqrt{1 - 2 \sin^2 \lambda}\, d\lambda, \tag{6}$$

and making the substitution $\sqrt{2} \sin \lambda = \sin \phi$, we have

$$c_1 x = \frac{1}{\sqrt{2}} \int \frac{\cos^2 \phi d\phi}{\sqrt{1 - \frac{1}{2} \sin^2 \phi}} + c_2. \tag{7}$$

We now apply the formula

$$\int_0^\phi \frac{\cos^2 \phi d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \frac{1}{k^2} [E(k, \phi) - k'^2 F(k, \phi)], \tag{8}$$

where $k^2 < 1$, $k'^2 = 1 - k^2$ and F and E are elliptic integrals of the first and second classes:

$$F(k, \phi) = \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}; \qquad E(k, \phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \phi}\, d\phi.$$

Writing $\sqrt{2} c_1 = a$ and $\sqrt{2} c_2 = b$, equation (7) becomes

$$ax = 2E\left(\frac{1}{\sqrt{2}}, \phi\right) - F\left(\frac{1}{\sqrt{2}}, \phi\right) + b.$$

We obtain y from the relation

$$dy = \tan \lambda\, dx.$$

Hence, from (6),

$$c_1 dy = \tan \lambda \sqrt{1 - 2 \sin^2 \lambda}\, d\lambda.$$

Making the same substitution as before, $\sqrt{2} \sin \lambda = \sin \phi$, and integrating, we obtain

$$c_1 y = \tan^{-1} \cos \phi - \cos \phi + c_3.$$

Writing as before $\sqrt{2} c_1 = a$, also $\sqrt{2} c_3 = c$, this becomes

$$ay = \sqrt{2}(\tan^{-1} \cos \phi - \cos \phi) + c.$$

We have, then, the solution of the system in parametric form:*

$$ax = 2E\left(\frac{1}{\sqrt{2}}, \phi\right) - F\left(\frac{1}{\sqrt{2}}, \phi\right) + b,$$
$$ay = \sqrt{2}\,(\tan^{-1}\cos\phi - \cos\phi) + c, \tag{9}$$

where a, b and c are arbitrary constants and ϕ is the parameter. For given values of a and b we have a family of ∞^1 curves admitting a group of translations along the y-axis. The variation of a and b gives ∞^2 such families.

As all of these curves may be obtained from any one of them by a translation and a homothetic transformation, there is essentially only one curve, which may be obtained, for example, by letting $a = 1$, $b = c = 0$, in equations (9)

The curve is symmetrical with respect to the y-axis and is periodic. It crosses the x-axis at an angle of 45° and is an elongated analogue of the sine curve.

Systems of Types V and VI Admitting a Group of Translations.

The substitution of $\lambda = \lambda(x)$ in equations V′ and VI′ leads to the same differential equation in both cases, namely

$$\lambda'' - \lambda'^2 \tan\lambda = 0 \tag{10}$$

or

$$y'''(1 + y'^2) = 3y'y''^2, \tag{11}$$

which is the differential equation of all circles in the plane. This system of ∞^3 circles is composed of ∞^2 families of the form $y' = \tan\lambda(x)$, each admitting a group of translations along the y-axis.

For any system of circles, $T = 0$, since the variation of the curvature of a circle along the circle is zero. In a family of ∞^1 circles obtained from a given circle by translation, $N = 0$, since the variation of the curvature of a circle of the family along an orthogonal curve is zero. The only solution of either $N + T = 0$ or $N - T = 0$ of the form $y' = \tan\lambda(x)$ is the simultaneous solution for which $T = N = 0$. This represents the totality of families of ∞^1 circles obtained by translating a circle in the direction of the y-axis.

Systems of Types VII and VIII Admitting a Group of Translations.

The substitution of $\lambda = \lambda(x)$ in equations VII′ and VIII′ gives in both cases

$$\lambda'^2 - \lambda'' + (\lambda'^2 + \lambda'')\tan\lambda = 0. \tag{12}$$

* An isothermal system (other than parallel straight lines) which admits a group of translations is of the form $y + \log\cos x = c$. See Kasner, loc. cit., p. 342.

The first integral of this equation is easily found to be

$$\lambda'(\cos\lambda - \sin\lambda) = c_1. \tag{13}$$

Integrating again, we find

$$c_1 x = \sin\lambda + \cos\lambda + c_2.$$

Also, from (13) and the relation

$$dy = \tan\lambda\, dx,$$

we find,

$$c_1 y = \sin\lambda - \cos\lambda - \log(\sec\lambda + \tan\lambda) + c_3.$$

Hence we have the equations of the system in parametric form:

$$\begin{aligned} ax &= \sin\lambda + \cos\lambda + b, \\ ay &= \sin\lambda - \cos\lambda - \log(\sec\lambda + \tan\lambda) + c. \end{aligned} \tag{14}$$

A Characteristic Geometric Property of the Complete System $T - T_1 = 0$. The vanishing of the function

$$T = \frac{d\kappa}{ds} = \frac{\lambda_{xx} + \lambda_x\lambda_y - (\lambda_x^2 - \lambda_y^2 - 2\lambda_{xy})y' + (\lambda_{yy} - \lambda_x\lambda_y)y'^2}{1 + y'^2} \tag{15}$$

at any point (x, y) of a curve of the system

$$y' = \tan\lambda(x, y) \tag{16}$$

is the condition that the curve is hyperosculated at this point by its circle of curvature, since the rate of variation of the curvature of the curve at this point is the same as that of a circle, namely zero.

Let us now consider a system of curves (16) and the system of ∞^1 isogonals through any point (x, y). The slope of an isogonal making an angle α with a curve of the given system (16) is

$$y' = \tan(\lambda + \alpha), \tag{17}$$

and the function T for this isogonal is of the same form (15) as for a curve of system (16), where y' now has the value given in equation (17). As α varies, equation (15) gives the value of T at the given point for each of the ∞^1 isogonals through this point. The function T will vanish for two of these isogonals whose directions are given by the roots of the equation

$$\lambda_{xx} + \lambda_x\lambda_y - (\lambda_x^2 - \lambda_y^2 - 2\lambda_{xy})y' + (\lambda_{yy} - \lambda_x\lambda_y)y'^2 = 0, \tag{18}$$

regarded as a quadratic in y', the coefficients being determined for the given point (x, y). The isogonals having these two directions are hyperosculated at the given point by their circles of curvature.

The condition that these two directions be perpendicular is

$$\lambda_{xx} + \lambda_{yy} = 0, \quad \text{or} \quad T + T_1 = 0,$$

which is the equation of an isothermal system. This is the characteristic geometric property of isothermal systems found by Kasner.*

We shall now obtain a characteristic property of the system, $T - T_1 = 0$, by considering its isogonal system. We wish to find the relation between the roots of equation (18) subject to the condition

$$T - T_1 = 0$$

or

$$\lambda_{xx} - \lambda_{yy} + 2\lambda_x\lambda_y - (\lambda_x^2 - \lambda_y^2 - 2\lambda_{xy}) \tan 2\lambda = 0. \tag{19}$$

Representing the roots of equation (18) by $\tan \theta_1$ and $\tan \theta_2$, we find

$$\frac{\tan \theta_1 \tan \theta_2 - 1}{\tan \theta_1 + \tan \theta_2} = \frac{\lambda_{xx} - \lambda_{yy} + 2\lambda_x\lambda_y}{\lambda_x^2 - \lambda_y^2 - 2\lambda_{xy}},$$

which, under condition (19), becomes

$$-\cot(\theta_1 + \theta_2) = \tan 2\lambda.$$

Hence the directions $\theta_1 + \theta_2$ and 2λ are orthogonal. Also, if the directions $\theta_1 + \theta_2$ and 2λ are orthogonal, the system is of the form (19). We have, then, a characteristic property of a system of curves of the form $T - T_1 = 0$:

The family of ∞^1 curves through a given point isogonal to a given simply infinite system, $y' = \tan \lambda(x, y)$, contains two isogonals which are hyperosculated by their circles of curvature. The directions θ_1 and θ_2 of these two isogonals are such that the direction $\theta_1 + \theta_2$ is orthogonal to the direction 2λ for every point of the plane when, and only when, the given system is of the form $T - T_1 = 0$.

COLUMBIA UNIVERSITY,
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* Bull. Am. Math. Soc., vol. 14, p. 170.

THE PROBABILITY OF THE ARITHMETIC MEAN COMPARED WITH THAT OF CERTAIN OTHER FUNCTIONS OF THE MEASUREMENTS.

By Edward L. Dodd.

1. Introduction.

In his Theorie der Beobachtungsfehler, Czuber has exhibited many of the attempts made to unite the principle of the arithmetic mean as the "most probable value" with the Gaussian probability law. He* quotes from Bertrand† who gives an example to show that this law and principle are not strictly compatible. It is one object of this paper to show this incompatibility by other methods,—to exhibit functions of the measurements to which the Gaussian law assigns a greater probability than it assigns to the arithmetic mean.‡

Wrapt up with the Gaussian law are several assumptions. Of these, we note the following in particular.

1. A true value, a, exists for the unknown.§

2. Associated with a measurement, m, or with a set of n measurements taken under similar circumstances, there exists a positive constant, h, called the measure|| of precision.

3. An objective¶ or physical probability may exist, when its value is unknown or but "approximately" known.

For example: if an urn contains just n balls of which w are white, the

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* Loc. cit., p. 51.

† Calcul des Probabilités (1889), p. 180.

‡ Sets of axioms have been proposed to ground the principle of the arithmetic mean as the "most probable value." See Czuber, loc. cit., p. 16–47; also Schimmack, Mathematische Annalen, 68. Band, p. 125. These axioms are, in general, of such a nature that they may be used equally well to ground the principle that the arithmetic mean is the least probable value.

§ This assumption and the next one presuppose that a unit of measure has been adopted. In changing from meters to centimeters, a would be multiplied by 100, h divided by 100, but ha would be invariant.

|| Here we merely set up the "assumption" or "axiom" that h exists. A commonly accepted approximation for h is $\sqrt{\dfrac{n-1}{2\Sigma v^2}}$, in which Σv^2 is the sum of the squares of the residuals of the measurements; i. e., $v_i = M - m_i$, M being the arithmetic mean. See Czuber, Wahrscheinlichkeitsrechnung, I, p. 281.

¶ For the view that probability is "purely subjective," see Sigwart's Logic, trans. by Dendy, vol. 2, p. 224. For a distinction between objective probability and subjective probability, see Kries, "Die Prinzipien der Wahrscheinlichkeitsrechnung (1886), p. 95. Reference is made to the distinction in the Encyklopädie (I, D. 1), p. 735.

probability of its delivering a white ball is w/n,—whether anyone knows what n and w are, or not. Note that h in 2. is unknown, as well as a in 1.

The Gaussian probability law then states that

$$P = \frac{h}{\sqrt{\pi}} \int_{x'}^{x''} e^{-h^2x^2} dx \tag{1}$$

is the probability that the error, $x = a - m$, will lie between x' and x'', where $x' \leqq x''$. The following, also, are important assumptions underlying (1), or inferences from (1), according to the viewpoint.

4. The probability, P, is a function of h, x', and x'', but not of a.*

This does not prevent the probability of the error of certain functions of the measurements being functions of a, h, x' and x''.

5. P is never zero when x' is less than x''. Roughly speaking: "Any error is possible."

6. P is zero if x' equals x''. "The probability of any particular error is zero."

7. P is unchanged, if $-x''$, $-x'$ replace x', x''. "The probability of a negative error is the same as that of the corresponding positive error."

8. P is greater for the interval $(-\alpha, +\alpha)$ than for any other interval of length, 2α. "Zero is the most probable error."

9. P is unity for the interval $(-\infty, +\infty)$. Thus the probability for very large errors is very small.

These assumptions, in general, are mathematical refinements, or abstractions,—somewhat comparable with the conception in geometry of a line with no breadth or thickness. However, it is not the object of this paper to defend these assumptions or to prove the Gaussian law, but to investigate its consequences.

From 6., it follows that under the Gaussian law, there is, strictly speaking, no most probable error. Each of the infinite number (see 5.) of possible errors has the same probability, zero.† For the discussion of the relative magnitude of probabilities, some definition is needed. Corresponding to each function, f, of the measurements, there is an error, $a - f$. When f lies in the interval from $a - \alpha$ to $a + \alpha$, its error lies in the interval from $-\alpha$ to α.

Definition.—The probability of f_1 will be said to be greater than that of f_2, if the probability that the error of f_1 will‡ lie in the interval from $-\alpha$ to $+\alpha$

* See Bertrand, loc. cit., p. 177.

† Similarly, the probability, a posteriori, that any particular real number be the true value is zero,—according to standard treatments. For example, see Poincaré, Calcul des Probabilités (1896), p. 149. Set $dz = 0$ in the numerator. There is here, then, no "most probable value" for the unknown true value.

‡ The future tense. By "measurements," as discussed here, will be meant contemplated measurements.

is greater than the probability that the error of f_2 will lie in the same interval, for all positive values of α less than some α',—in other words, if the probability that f_1 will differ from a by less than α is greater than that f_2 will differ from a by less than α, when $\alpha < \alpha'$.

Some such restriction as that imposed upon α is needed to avoid anomalies. For example: if α is taken equal to a, the probability of bm is greater than that of m, where b is any proper fraction. For bm will lie in the interval, $(a - \alpha,\ a + \alpha) = (0,\ 2a)$, whenever m lies in $(0,\ 2a/b)$. Furthermore, it is natural to make α small; for, in general, the error of each measurement will be small in comparison with a. But α cannot be made zero; for the probability that f_1 or f_2 will equal a is zero.* For example: the probability that the average of four measurements will differ from a by less than α, is found by replacing h by $2h$, and x', x'', by $-\alpha$, α in (1). This probability vanishes with α.

2. The Comparative Probabilities of the Arithmetic Mean of Measurements and the Square Root of the Mean of Their Squares.

Under the Gaussian law (1), it follows that the probability, P_α, that the average, M, of two measurements will lie in the interval, $(a-\alpha,\ a+\alpha)$, is given by

$$P_\alpha = \frac{2\sqrt{2}h}{\sqrt{\pi}}\int_0^\alpha e^{-2h^2x^2}dx = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{2}h\alpha} e^{-t^2}dt = \Theta(\sqrt{2}h\alpha). \tag{2}$$

This is not a new theorem. It may be proved as follows.† Let the error of m_1 be x, and that of m_2 be y. P_α is then the probability that

$$-\alpha \leqq a - \frac{m_1 + m_2}{2} \leqq \alpha,$$

or that

$$\begin{aligned} -2\alpha &\leqq x + y \leqq 2\alpha, \\ -2\alpha - x &\leqq y \leqq 2\alpha - x. \end{aligned} \tag{3}$$

The first error, x, may be of any magnitude; but the second, y, must then satisfy (3). Hence,

$$P_\alpha = \frac{h}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-h^2x^2}dx \frac{h}{\sqrt{\pi}}\int_{-2\alpha-x}^{2\alpha-x} e^{-h^2y^2}dy.$$

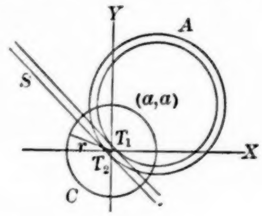
* Except for some trivial function, as $0m + a$.

† The proof given here is regarded as simpler than a proof using a non-convergent "discontinuity factor," such as

$$\int_{-\infty}^{\infty} \cos(\kappa\theta)d\theta \quad \text{or} \quad \int_{-\infty}^{\infty} e^{\kappa\theta\sqrt{-1}}\,d\theta.$$

Likewise, it may be proved that the probability that the error, E, of M will lie in $(0,\ \alpha)$ is $\frac{1}{2}\Theta(\sqrt{2}h\alpha)$. Thus the probability that E will lie in (x', x'') is given by (1) with h changed to $\sqrt{2}h$.

The field of integration, S, is bounded by two parallel lines, and its width is $2\sqrt{2}\alpha$. If the axes are rotated through 45°, the integrand is unchanged, but the boundaries of the field become parallel to the Y axis, at a distance of $\sqrt{2}\alpha$ from it. This gives (2).



Now let

$$\sqrt{\frac{m_1^2 + m_2^2 + \cdots + m_n^2}{n}}$$

be called the root-mean-square of the n measurements.

THEOREM 1. *Under the Gaussian probability law, the root-mean-square of two measurements has a greater probability than their arithmetic mean, provided the product of the precision constant by the true value is greater than* 2.

Proof.—The condition mentioned is

$$ha > 2. \tag{4}$$

Now h is positive, and thus a is also.

Let P_α' be the probability that

$$a - \alpha \leqq \sqrt{\frac{m_1^2 + m_2^2}{2}} \leqq a + \alpha.$$

The errors being x and y as before, x and y must satisfy

$$2(a - \alpha)^2 \leqq (x - a)^2 + (y - a)^2 \leqq 2(a + \alpha)^2; \tag{5}$$

in this, α is to be taken less than a. The point, (x, y), is then confined to an annular region, A, bounded by two concentric circles, with centers at (a, a) and radii, $\sqrt{2}(a - \alpha)$ and $\sqrt{2}(a + \alpha)$, respectively. The width of the ring is $2\sqrt{2}\alpha$.

Then

$$P_\alpha' = \frac{h^2}{\pi} \int_A \int e^{-h^2(x^2+y^2)} dx dy, \tag{6}$$

whereas P_α is a like integral taken over the strip, S, of the same width, $2\sqrt{2}\alpha$. If the integrand is set equal to z, the locus is a surface of revolution about the Z axis, and thus the integrand takes the same value for every point on a circle, C, of radius, r, in the XY plane, centered at the origin. Consider now the evaluation of P_α' and P_α, under a transformation to polar coördinates. If

$$\sqrt{2}\alpha < r < \sqrt{2}(2a - \alpha),$$

arcs of the circle, C, will be intercepted between the boundaries of S and also of A; but the latter arcs will be the greater, because their chords are greater,—a straight line segment joining a point on one boundary of A

to a point on the other boundary will be greater than $2\sqrt{2}\alpha$, unless it is a portion of the radius of the outer circle. Now, usually the integrand in (6) will become inappreciable long before r reaches $2\sqrt{2}a$; usually, errors double the true value are well-nigh impossible. The infinite portions of S contribute next to nothing—when $ha > 2$—to the integral, P_a; whereas P_a' outstrips P_a in the stretch from 0 to $2\sqrt{2}a$. To get a numerical relation between P_a and P_a', we may proceed as follows:

Let the axes be rotated through 45°; and set $R = a\sqrt{2}$, $\delta = \alpha\sqrt{2}$. Let the intersections of C with the boundaries of A in the first quadrant be (x_1, y_1) and (x_2, y_2) when

$$\delta < r < 2R - \delta. \tag{7}$$

Find the value of $y_2 - y_1$ and to it apply the inequality,

$$\sqrt{c+d} - \sqrt{c-d} > d/\sqrt{c}, \qquad \text{if } 0 < d < c.$$

Then, if D is the length of the chord,

$$D^2 > \frac{4\delta^2}{1 - \dfrac{r^2 - \delta^2}{4R^2}},$$

$$D > 2\delta\left[1 + \frac{r^2 - \delta^2}{8R^2}\right].$$

Then if θ is the angle formed by straight lines from the origin to (x_1, y_1) and (x_2, y_2), respectively,

$$\theta > \frac{2\delta}{r}\left[1 + \frac{r^2 - \delta^2}{8R^2}\right]. \tag{8}$$

Now, if (7) did not need to be satisfied, it would be found upon passing to polar coördinates, and using

$$\frac{2h}{\sqrt{\pi}}\int_0^\infty e^{-h^2r^2}dr = 1, \qquad \frac{2h}{\sqrt{\pi}}\int_0^\infty e^{-h^2r^2}r^2dr = \frac{1}{2h^2},$$

that P_a' would—when δ is small—be greater than

$$\frac{2\delta h}{\sqrt{\pi}}\left[1 + \frac{1}{32h^2a^2}\right]. \tag{9}$$

But, because (8) was obtained from (7) with $r < 2R - \delta$, a deduction must be made from the bracket in (9) of an amount not greater than $3/[80(ha)^3]$, together with an amount which approaches zero with δ,—

because of the inequality, $\delta < r$, in (7). The upper limit mentioned for the former deduction is obtained by using the inequalities,

$$\int_{2R}^{\infty} e^{-h^2r^2}dr = \frac{1}{h}\int_{2hR}^{\infty} e^{-t^2}dt < \frac{1}{2h^2R}\int_{2hR}^{\infty} e^{-t^2}tdt,$$

$$\int_{2R}^{\infty} e^{-h^2r^2}r^2dr < \frac{1}{2R}\int_{2R}^{\infty} e^{-h^2r^2}r^3dr.$$

But

$$P_a < \frac{2\delta h}{\sqrt{\pi}};$$

and hence if $ha > 2$, and δ sufficiently small,

$$P_a' > P_a + \frac{2\delta h}{\sqrt{\pi}}\left[\frac{1}{80(ha)^2}\right]. \qquad \text{Q.E.D.}$$

It is not to be supposed that 2 is the critical value for ha. But it is evident from (6) that, for a given a, it would be possible to choose an h so small that the integrand in (6) would be sensibly unity throughout A and the nearer portions of S. The remote portions of S would then make P_a greater than P_a'.

THEOREM 2. *Under the Gaussian law, the probability of the arithmetic mean of three measurements is greater than the probability of their root-mean-square.*

Proof.—In this case, the probability, P_a, that the arithmetic mean will differ from the true value by less than α, is $\Theta(\sqrt{3}h\alpha)$. It may be found as a triple integral over a region bounded by parallel planes, each at a distance, $\sqrt{3}\alpha$, from the origin,—analogous to S in the figure. Likewise, for three measurements, a being positive, P_a' is the integral over a region between two concentric spheres, centered at (a, a, a), and tangent to the two planes bounding the new S. The zones of the sphere, C, cut off by the regions, S and A, have the same altitude and thus the same area. On these zones —for fixed r—the integrand is a constant, and each of the two zones gives the same integral. But as r goes from zero toward infinity, the region A becomes exhausted, whereas S does not. Hence $P_a' < P_a$. But, in general, their difference would be inappreciable. No condition, such as $a > 0$, is needed in dealing with three measurements. For, in case $a = 0$, the region A becomes a sphere lying in S. And if $a < 0$, P_a' is zero for a small α,—if the usual convention, giving to the radical the positive sign, is adopted. But if the radical is to be made always negative, the treatment is essentially that for $a > 0$.

When n is very large, a certain presumption exists that the arithmetic

mean, M, has a greater probability than the root-mean-square, M'. For if M is positive, and $v_1, v_2, \cdots$ are the residuals, then

$$M' = M\left[1 + \frac{\Sigma v^2}{nM^2}\right]^{\frac{1}{2}}.$$

Now the arithmetic mean is subject to the Gaussian law with precision constant, $\sqrt{n}h$. And it will be proved presently that the probability of bM is less than that of M, when b is a constant greater than unity. When n is very large, the bracket above is supposed to be sensibly a constant, and it is greater than unity.

That the arithmetic mean, M, of n measurements,—each subject to the Gaussian law with precision, h,—is subject to this law, with precision $\sqrt{n}h$, may be proved as follows. The condition that the error of M shall lie in $(-\alpha, \alpha)$, is equivalent to the condition that the sum, Σx, of the errors of the measurements shall lie in $(-n\alpha, n\alpha)$. The probability for this is an n-fold integral taken over a region bounded by two " parallel planes " in n dimensions. Each plane is at a " distance," $\sqrt{n}\alpha$ from the origin. By an orthogonal transformation—" rotation "—the planes can be made " perpendicular " to an " axis." The following is such a transformation:

$$X_1 = \frac{x_1}{\sqrt{n}} + \frac{x_2}{\sqrt{n}} + \frac{x_3}{\sqrt{n}} + \cdots + \frac{x_n}{\sqrt{n}},$$

$$X_2 = \frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}},$$

$$X_3 = \frac{x_1}{\sqrt{2 \cdot 3}} + \frac{x_2}{\sqrt{2 \cdot 3}} - \frac{2x_3}{\sqrt{2 \cdot 3}},$$

$$X_r = \frac{x_1}{\sqrt{(r-1)r}} + \frac{x_2}{\sqrt{(r-1)r}} + \cdots + \frac{x_{r-1}}{\sqrt{(r-1)r}} - \frac{(r-1)x_r}{\sqrt{(r-1)r}},$$

where $2 \leqq r \leqq n$. Then $X_1 = \Sigma x/\sqrt{n}$, and lies in $(-\sqrt{n}\alpha, \sqrt{n}\alpha)$.

3. The Comparative Probabilities of m and bm, and also of M and bM.

Since the average, M, of n measurements, is subject to the Gaussian probability law, with precision constant, $\sqrt{n}h$, a comparison of the probabilities of m and bm—where b is a constant—is likewise a comparison of the probabilities of M and bM,—with the proper change from h to $\sqrt{n}h$.

THEOREM 3. *Under the Gaussian law, the probability of bm is less than that of m, if b is a constant greater than unity; but there exist positive values of the constant b for which the probability of bm is greater than that of m.*

Proof.—Let P_a be the probability that m will differ from a by less than α, and let P_a' be the probability that

$$a - \alpha \leqq bm \leqq a + \alpha,$$

or

$$-\frac{a+\alpha}{b} \leqq -m \leqq -\frac{a-\alpha}{b},$$

$$a - \frac{a+\alpha}{b} \leqq a - m \leqq a - \frac{a-\alpha}{b}.$$

By hypothesis, this probability is

$$P_a' = \frac{h}{\sqrt{\pi}} \int_{x'}^{x''} e^{-h^2x^2}\, dx, \tag{10}$$

where

$$x' = a - \frac{a+\alpha}{b}, \qquad x'' = a - \frac{a-\alpha}{b}.$$

The interval of integration is, in length, $2\alpha/b$. In the special case, where $a = 0$, $P_a' > P_a$ if $0 < b < 1$; but $P_a' < P_a$ if $b > 1$. Also, in all other cases,

$$P_a' < P_a, \qquad \text{if } b > 1.$$

For the interval of integration for P_a' is smaller, and is less favorably situated—being centered at $a - a/b$. Likewise, when $0 < b < 1$, the center of the interval is at $a - a/b$; but the length of the interval is greater than 2α, the interval for P_a. It will now be shown that if b is taken sufficiently close to unity—thus bringing the center of the interval close to the origin—the advantage which P_a' has in length of interval, outweighs the disadvantage in position; i. e., $P_a' > P_a$.

Now the integrand is greatest when $x = 0$. Hence

$$P_a < \frac{2h\alpha}{\sqrt{\pi}}.$$

On the other hand, if $a > 0$, the integrand for P_a' is least when $x = a - (a + \alpha)/b$. Thus

$$P_a' > \frac{2\alpha}{b} \frac{h}{\sqrt{\pi}} e^{-h^2[a(1-b)+\alpha]^2/b^2}.$$

Hence $P_a' > P_a$, provided b and α can be so taken that

$$\frac{1}{b} e^{-h^2[a(1-b)+\alpha]^2/b^2} > 1,$$

that is,

$$\frac{h^2}{b^2}[a(1-b)+\alpha]^2 < \log_e \frac{1}{b}.$$

Take $\alpha \leqq a(1 - b)$, and let $1/b = 1 + y$. It is then to be shown that y can be taken positive but small enough so that

$$4h^2a^2y^2 < \log_e (1 + y).$$

But when y is sufficiently small,

$$4h^2a^2y^2 < y - \frac{y^2}{2} < \log_e (1 + y). \qquad \text{Q.E.D.}$$

The proof for the case, $a < 0$, is essentially the same.

Example.—If the Gaussian law be assumed, and if it be admitted that of two values for the unknown it is better to accept that which has the greater probability, in accordance with the definition adopted in this paper, then the foregoing theorem implies that if a meter bar is measured in inches with the result, 39.37, it is better to accept for the length of the bar some number a little less than 39.37 than to accept 39.37 itself. This applies whether 39.37 is a single measurement or is the arithmetic mean of a set of measurements taken under the same circumstances,—for the arithmetic mean is subject to the Gaussian law when the individual measurements are thus subject.

But the difference of 39.37 and such a number is so small—if the measurements have been made with even a moderate degree of accuracy—as to be negligible. This may be seen as follows. With α small, the integral (10), as a function of b, has a maximum for approximately

$$b = 1 - \frac{1}{2h^2a^2}. \tag{11}$$

Furthermore, $P_\alpha' > P_\alpha$ if

$$1 > b > 1 - \frac{2}{2h^2a^2 + 3}, \tag{12}$$

and α is small enough. Or, if the arithmetic mean of n measurements is used, then in place of (12), we have

$$1 > b > 1 - \frac{2}{2nh^2a^2 + 3}. \tag{13}$$

But, ordinarily, h^2a^2 is very large, and this multiplier, b, to be used upon m or upon the arithmetic mean, M, does not differ appreciably from unity. Indeed, in the case of the meter bar, it may be reasonably certain that $a > 39$. Now the commonly accepted formula* connecting h with the so-called "probable error," r, is

$$hr = .476936.$$

* Czuber, Wahrscheinlichkeitsrechnung, I, p. 270.

To take $r = .01$ signifies that we suppose it as likely that the error of a measurement will be numerically less than a hundredth of an inch as that it will exceed that amount. In this case, $h > 47$; and, hence,

$$h^2a^2 > 3{,}000{,}000.$$

A modification of the foregoing theorem consists in the use of a function, $bm + c$, with $b < 1$ and $c > 0$,—provided $a > 0$. The c does not affect the length of the interval of integration, but merely its position. With b fixed, there exist positive values of c which move the interval toward the origin and thus augment the probability.

THEOREM 4. *Under the Gaussian probability law, the probability of* $m + c$ *is less than that of* m, *for every value of the constant,* c, *not zero.*

The proof of this theorem follows at once from considerations given above.

4. The Probability of the Median.

By the median of $2\nu + 1$ measurements is meant the middle or $(\nu+1)$th measurement when they are arranged according to magnitude. If there are three measurements, the first or second or third measurement made —in order of time—may be the median. If the first is the median, the measurement less than the median may be the second, or it may be the third. The probability that the error of m_1 will lie in $(-\alpha, \alpha)$ is very nearly $2h\alpha/\sqrt{\pi}$ when α is small; the probability that m_2 will then be less than m_1 is nearly $1/2$; and the probability that m_3 will then be greater than m_1 is nearly $1/2$. There being six arrangements of the m's, the probability of the median is nearly $3h\alpha/\sqrt{\pi}$. This is about 87 per cent of the approximate probability, $2\sqrt{3}h\alpha/\sqrt{\pi}$, of the arithmetic mean. The exact probability, —according to the Gaussian law—that the median of $2\nu + 1$ measurements will lie in $(a - \alpha, a + \alpha)$ is

$$P_\alpha' = \frac{(2\nu+1)!}{4^\nu(\nu!)^2}\frac{h}{\sqrt{\pi}}\int_{-\alpha}^{\alpha}[1 - \Theta^2(hx)]^\nu e^{-h^2x^2}dx. \tag{14}$$

In this expression, Θ and thus its square become negligible when, for any given ν, α is made sufficiently small.

By Stirling's formula,*

$$1 \cdot 2 \cdot 3 \cdots \nu = \nu! = \sqrt{2\pi\nu}\,\nu^\nu e^{-\nu+\theta/12\nu}, \qquad 0 < \theta < 1.$$

Hence, if P_α is the probability that the arithmetic mean will lie in $(a - \alpha, a + \alpha)$,

* See, for example, Broggi, Traité des Assurances sur la Vie (1907), p. 54.

$$\frac{P_a'}{P_a} = \sqrt{\frac{2\nu + 1}{\pi\nu}}\, e^{\theta\, 24\nu - \theta' 6\nu}(1 + \epsilon),$$

where $\lim_{a=0} \epsilon = 0,\ 0 < \theta < 1,\ 0 < \theta' < 1.$

THEOREM 5. *The probability of the median of an odd number, $2\nu + 1$, of measurements is less than that of their arithmetic mean—under the Gaussian law—and if ν is made sufficiently large, and then α taken small enough, the ratio of P_a' to P_a can be made as near $\sqrt{2}/\sqrt{\pi} = .7979$ as we please.*

Thus, with a large number of measurements, the probability of the median falls about 20 per cent short* of that of the arithmetic mean.

5. The Probability of the Geometric Mean.

By the geometric mean of two positive measurements, m_1 and m_2, will be meant $+\sqrt{m_1 m_2}$; but the negative radical will be taken if both are negative; and the mean will not be regarded as defined, if they have unlike signs. Likewise, for n measurements, the geometric mean will be regarded as positive, if all measurements are positive; negative, if all measurements are negative; otherwise, undefined.

THEOREM 6. *The probability of the geometric mean is less than that of the arithmetic mean, under the Gaussian law.*

This can be proved for the three cases: $a > 0$, $a = 0$, $a < 0$. In the case of two measurements, when $a > 0$, the field of integration for the geometric mean is bounded by two equilateral hyperbolas, tangent to the boundaries of S—see figure—at T_1 and T_2; and extending out the second quadrant, and down the fourth quadrant, asymptotic to the lines, $y = a$, and $x = a$. The segment, T_1T_2, is the only straight line segment with slope, unity, joining the two hyperbolas, and having a length as great as $2\sqrt{2}\alpha$. If the axes are rotated through 45°, and then integration is performed first with respect to x; the integral with y fixed and not zero—will be less for the geometric mean than for the arithmetic mean. In n dimensions, the proof is facilitated, by translating to $(a, a \cdots a)$ as a new origin, using the "surfaces" upon which the points have positive coordinates, and showing that if $(x_1, x_2, \cdots x_n)$ lies on one surface, $(x_1 + 2\alpha, x_2 + 2\alpha, \cdots x_n + 2\alpha)$ falls "outside" the region bounded by the two surfaces.

6. The Probability of the Weighted Mean.

By a weighted mean of n measurements is meant,

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* This does not necessarily discredit the use of the median in economic, biological or other investigations. Only Gaussian distributions are being considered in this article.

$$p_1m_1 + p_2m_2 + \cdots + p_nm_n,$$

where the p's are given constants such that

$$p_1 + p_2 + \cdots + p_n = 1.$$

THEOREM 7. *If n measurements are subject to the Gaussian law, with precision constants, $h_1, h_2, \cdots h_n$, respectively, then any weighted mean is also subject to the Gaussian law,* with precision constant, H, where*

$$\frac{1}{H^2} = \Sigma\left(\frac{p}{h}\right)^2.$$

This H takes its greatest value when

$$p_1 = \frac{h_1^2}{\Sigma h^2}, \quad p_2 = \frac{h_2^2}{\Sigma h^2}, \quad \cdots, \quad p_n = \frac{h_n^2}{\Sigma h^2};$$

and in this case,

$$H^2 = \Sigma h^2;$$

and the probability of this weighted mean is greater than that of any other weighted mean.

The proof of this theorem † involves the use of an orthogonal transformation which can be set up with zero coefficients in the same places as in the orthogonal transformation indicated for the arithmetic mean. Now H has a maximum when $\Sigma(p/h)^2$ has a minimum. The case, $p_n = 0$, can be considered separately. Otherwise,

$$p_n = 1 - p_1 - p_2 - \cdots - p_{n-1}$$

can be inserted, and the minimum located by setting the first partial derivatives equal to zero. For this point, an actual minimum and least value will occur; since

$$\Delta\Sigma\left(\frac{p}{h}\right)^2 = \sum_1^{n-1}\left(\frac{\Delta p}{h}\right)^2 + \frac{1}{h_n^2}\left(\sum_1^{n-1}\Delta p\right)^2.$$

It should be noted that this weighted mean has not been proved to be the "most probable value." In fact, since this weighted mean, W, is subject to the Gaussian law, there are constants, b, a little less than unity, —as has been proved in Theorem 3—such that bW has a greater probability than W.

* Cf. Czuber, Wahrscheinlichkeitsrechnung, I, p. 260.

† For a generalization of this theorem, see: Dodd, "The Least Square Method grounded with the aid of an Orthogonal Transformation," Jahresbericht der Deutschen Mathematiker-Vereinigung, 21. Band (1912), p. 177.

Corollary.—The probability of the arithmetic mean of n measurements with the same precision, h, is greater than the probability of any other linear homogeneous function of the measurements with constant coefficients whose sum is unity.

For, in the first place, the arithmetic mean is the weighted mean for which each weight is $1/n$. This weight, $1/n$, is the most favorable weight for each measurement, when all the h's are equal. The general formula,

$$H^2 = \Sigma h^2,$$

becomes in this case,

$$H^2 = nh^2$$

as found before.

CUSP AND UNDULATION INVARIANTS OF RATIONAL CURVES.*

By J. E. Rowe.

Introduction.

It would be useful to have a formal statement of a method by means of which the cusp and undulation conditions of a rational plane curve could be written down in their best forms. These invariants are expressible in terms of the three-rowed determinants of the matrix of coefficients of the parametric equations of the curve. Meyer† has shown that the cusp condition is of order $2(n-1)$ and the undulation condition of order $4(n-3)$ in these determinants but it would be difficult to actually write them out by using his methods. Also, by the use of perspective curves, introduced by Stahl,‡ it is possible to write out the cusp condition as a determinant of order $6(n-1)$ but the high order of this determinant makes the method of little practical use. It is the purpose of this paper to give a straightforward method of writing out these invariants for a rational curve of order n as determinants of orders $2(n-1)$ and $4(n-3)$ whose constituents are the three-rowed determinants mentioned above. These methods are valuable because they may be directly extended to find the corresponding singularities in higher dimensions. Other interesting facts that are brought out in the paper are the peculiar relation of the cusp and undulation conditions of the plane rational quintic which is a special case of a relation peculiar to all invariants of the plane rational quintic, and the generalization of these facts in higher dimensions.

The Undulation Invariants.

Let the parametric equations of the rational curve of order n in space of d dimensions which we call $R_d{}^n$ be written

$$(1)\qquad x_i = a_i t^n + nb_i t^{n-1} + \frac{n(n-1)}{1-2} c_i t^{n-2} \cdots; \qquad i = 0, 1, 2, 3, \cdots d;$$

also, when symbolic expressions are used for the n-ics on the right side of (1) it is to be understood that (1) is written in the form

$$(2)\qquad x_0 = (\alpha t)^n, \quad x_1 = (\beta t)^n, \quad x_2 = (\gamma t)^n, \quad \cdots x_d = (\pi t)^n.$$

* Presented to the American Mathematical Society, February 24, 1912.

† Meyer, Mathematische Annalen, vol. 38 (1891), pp. 375–6.

‡ W. Stahl, Mathematische Annalen, vol. 38 (1891), pp. 561–85.

Beginning with the plane we observe that any line

$$(\xi x) = \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2 = 0 \tag{3}$$

cuts (2) in n points whose parameters are the roots of the binary n-ic

$$\xi_0(\alpha t)^n + \xi_1(\beta t)^n + \xi_2(\gamma t)^n = 0. \tag{4}$$

There are $3n - 6$ lines which cut (2) in three consecutive parameters; these $(3n - 6)$ parameters are the flex parameters of the $R_2{}^n$, and are the values of t which occur as cubed factors in members of the system of binary n-ics (4). There is no member of (4) which contains a fourfold factor unless a certain relation exists among the coefficients of the parametric equations of the $R_2{}^n$. Translating the theory of partial derivatives into symbolic notation* we find that the condition for the system (4) to have a member which contains a fourfold factor is expressed by

$$(\alpha t)^{n-3}(\beta t)^{n-3}(\gamma t)^{n-3} \begin{vmatrix} \alpha_1{}^3 & \alpha_1{}^2\alpha_2 & \alpha_1\alpha_2{}^2 & \alpha_2{}^3 \\ \beta_1{}^3 & \beta_1{}^2\beta_2 & \beta_1\beta_2{}^2 & \beta_2{}^3 \\ \gamma_1{}^3 & \gamma_1{}^2\gamma_2 & \gamma_1\gamma_2{}^2 & \gamma_2{}^3 \end{vmatrix} = 0; \tag{5}$$

that is, if such a value of t exists it must satisfy simultaneously the four $(3n - 9)$-ics (5) obtained by dropping out in succession one column in the matrix. The coefficients of these $(3n - 9)$-ics are expressible in terms of the three-rowed determinants of the matrix of coefficients of the parametric equations of the $R_2{}^n$. The condition for t to satisfy these four equations simultaneously can be obtained as a determinant of order $4(n - 3)$ in their coefficients. Several illustrations will make this clear. In as much as calculation is an essential feature of the paper it will be necessary to write out explicitly more than the usual number of expressions. Take the $R_2{}^4$ as an example. From (1) and (2) we have

$$R_2{}^4 = x_i = a_i t^4 + 4b_i t^3 + 6c_i t^2 + 4d_i t + e_i; \qquad i = 0, 1, 2; \tag{6}$$

and

$$x_0 = (\alpha t)^4, \quad x_1 = (\beta t)^4, \quad x_2 = (\gamma t)^4. \tag{7}$$

For (7) the expression (5) becomes

$$(\alpha t)(\beta t)(\gamma t) \begin{vmatrix} \alpha_1{}^3 & \alpha_1{}^2\alpha_2 & \alpha_1\alpha_2{}^2 & \alpha_2{}^3 \\ \beta_1{}^3 & \beta_1{}^2\beta_2 & \beta_1\beta_2{}^2 & \beta_2{}^3 \\ \gamma_1{}^3 & \gamma_1{}^2\gamma_2 & \gamma_1\gamma_2{}^2 & \gamma_2{}^3 \end{vmatrix} = 0. \tag{8}$$

Let us actually calculate the two cubics found by striking out the fourth and third columns in the matrix (8); the first of these we shall refer to as (9) and the second as (10). Further

* See Grace & Young, Algebra of Invariants, p. 9.

$$
\begin{aligned}
(\alpha t)(\beta t)(\gamma t) &= (\alpha_1 t + \alpha_2)(\beta_1 t + \beta_2)(\gamma_1 t + \gamma_2) \\
&= \alpha_1\beta_1\gamma_1 t^3 + (\alpha_1\beta_1\gamma_2 + \alpha_1\beta_2\gamma_1 + \alpha_2\beta_1\gamma_1)t^2 \\
&\qquad + (\alpha_1\beta_2\gamma_2 + \alpha_2\beta_1\gamma_2 + \alpha_2\beta_2\gamma_1)t + \alpha_2\beta_2\gamma_2.
\end{aligned} \tag{11}
$$

Substituting in (9) and (10) from (11) and then from (6) in (9) and (10) we have

$$
\begin{aligned}
&\begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} t^3 + \left[\begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ b_2 & c_2 & d_2 \end{vmatrix} + \begin{vmatrix} a_0 & b_0 & c_0 \\ b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 \end{vmatrix} + \begin{vmatrix} b_0 & c_0 & d_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \right] t^2 \\
&+ \left[\begin{vmatrix} a_0 & b_0 & c_0 \\ b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{vmatrix} + \begin{vmatrix} b_0 & c_0 & d_0 \\ a_1 & b_1 & c_1 \\ b_2 & c_2 & d_2 \end{vmatrix} + \begin{vmatrix} b_0 & c_0 & d_0 \\ b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \right] t + \begin{vmatrix} b_0 & c_0 & d_0 \\ b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \end{vmatrix} = 0,
\end{aligned} \tag{12}
$$

and

$$
\begin{aligned}
&\begin{vmatrix} a_0 & b_0 & d_0 \\ a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \end{vmatrix} t^3 + \left[\begin{vmatrix} a_0 & b_0 & d_0 \\ a_1 & b_1 & d_1 \\ b_2 & c_2 & e_2 \end{vmatrix} + \begin{vmatrix} a_0 & b_0 & d_0 \\ b_1 & c_1 & e_1 \\ a_2 & b_2 & d_2 \end{vmatrix} + \begin{vmatrix} b_0 & c_0 & e_0 \\ a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \end{vmatrix} \right] t^2 \\
&+ \left[\begin{vmatrix} a_0 & b_0 & d_0 \\ b_1 & c_1 & e_1 \\ b_2 & c_2 & e_2 \end{vmatrix} + \begin{vmatrix} b_0 & c_0 & e_0 \\ a_1 & b_1 & d_1 \\ b_2 & c_2 & e_2 \end{vmatrix} + \begin{vmatrix} b_0 & c_0 & e_0 \\ b_1 & c_1 & e_1 \\ a_2 & b_2 & d_2 \end{vmatrix} \right] t + \begin{vmatrix} b_0 & c_0 & e_0 \\ b_1 & c_1 & e_1 \\ b_2 & c_2 & e_2 \end{vmatrix} = 0,
\end{aligned} \tag{13}
$$

which are the cubics (9) and (10) in expanded form; the coefficients of (12) and (13) are expressible in terms of the three-rowed determinants of the matrix of coefficients of (6); we obtain

$$|abc|t^3 + |abd|t^2 + |acd|t + |bcd| = 0, \tag{12'}$$

$$|abd|t^3 + [|abe| + |acd|]t^2 + [|ace| + |bcd|]t + |bce| = 0; \tag{13'}$$

the other two cubics of (8) can be written out without actual calculation because of their symmetry with respect to (13′) and (12′); they are

$$|acd|t^3 + [|ace| + |bcd|]t^2 + [|ade| + |bce|]t + bde = 0, \tag{14}$$

$$|bcd|t^3 + |bce|t^2 + |bde|t + |cde| = 0. \tag{15}$$

Any three of these equations (12′), (13′), (14), and (15) could be solved for t^3, t^2, and t, and these values substituted in the fourth yields the condition* for there to be a value of t which satisfies the four simultaneously; that is

* It is better to say that this yields the condition for a value of t which satisfies three of the equations to satisfy the fourth.

the determinant of the coefficients of the four equations (12′), (13′), (14), and (15) which is

$$
(16)\qquad \begin{vmatrix} |abc|, & |abd|, & |acd|, & |bcd| \\ |abd|, & |abe|+|acd|, & |ace|+|bcd|, & |bce| \\ |acd|, & |ace|+|bcd|, & |ade|+|bce|, & |bde| \\ |bcd|, & |bce|, & |bde|, & |cde| \end{vmatrix} = 0
$$

is the condition for the R_2^4 to have an undulation. This* invariant for the R_2^4 has been found before, but it is given here as an illustration of the general method.

The Special Case of the R_2^5.

From equations (1) and (2) the R_2^5 may be written

$$
(17)\qquad x_i = a_i t^5 + 5b_i t^4 + 10c_i t^3 + 10d_i t^2 + 5e_i t + t_i, \qquad i = 0, 1, 2,
$$

or

$$
(18)\qquad \begin{aligned} x_0 &= (\alpha t)^5 \\ x_1 &= (\beta t)^5 \\ x_2 &= (\gamma t)^5. \end{aligned}
$$

The matrix corresponding to (5) in this case is

$$
(19)\qquad (\alpha t)^2(\beta t)^2(\gamma t)^2 \begin{vmatrix} \alpha_1^3 & \alpha_1^2\alpha_2 & \alpha_1\alpha_2^2 & \alpha_2^3 \\ \beta_1^3 & \beta_1^2\beta_2 & \beta_1\beta_2^2 & \beta_2^3 \\ \gamma_1^3 & \gamma_1^2\gamma_2 & \gamma_1\gamma_2^2 & \gamma_2^3 \end{vmatrix} = 0.
$$

For convenience let the following system of abbreviations be used for the R_2^5:

$$
(20)\qquad \begin{array}{lllll} \alpha = |abc| & \beta = |abd| & \gamma = |abe| & \delta = |abf| & \lambda = |acd| \\ \alpha' = |def| & \beta' = |cef| & \gamma' = |bef| & \delta' = |aef| & \lambda' = |cdf| \\ \mu = |ace| & \nu = |acf| & \psi = |ade| & \varphi = |bcd| & \chi = |bce| \\ \mu' = |bdf| & \nu' = |adf| & \psi' = |bcf| & \varphi' = |cde| & \chi' = |bde|. \end{array}
$$

The four sextics (19) may now be put in the form

$$
(21)\qquad \begin{aligned} &\alpha t^6 + 2\beta t^5 + (\gamma + 3\lambda)t^4 + (2\mu + 4\varphi)t^3 + (\psi + 3\chi)t^2 + 2\chi' t + \varphi' = 0, \\ &\beta t^6 + 2(\gamma + \lambda)t^5 + (\delta + 4\mu + 3\varphi)t^4 + (2\nu + 2\psi + 6\chi)t^3 \\ &\qquad + (\nu' + 4\chi' + 3\psi')t^2 + 2(\varphi' + \mu')t + \lambda' = 0, \\ &\lambda t^6 + (2\varphi + \mu)t^5 + (\nu + 4\chi + 3\psi)t^4 + (2\nu' + 2\psi' + 6\chi')t^3 \\ &\qquad + (\delta' + 4\mu' + 3\varphi')t^2 + 2(\gamma' + \lambda')t + \beta' = 0, \\ &\phi t^6 + 2\chi t^5 + (\psi' + 3\chi')t^4 + (2\mu' + 4\varphi')t^3 + (\gamma' + 3\lambda')t^2 \\ &\qquad + 2\beta' t + \alpha' = 0. \end{aligned}
$$

* See Transactions of the American Mathematical Society, vol. 12, No. 3, p. 304.

If these four equations are multiplied by t the four equations so obtained together with the four of (21) form 8 septemics whose determinant equated to zero is the required condition. Hence the undulation of the R_2^5 of (17) is

$$(22)\qquad \begin{vmatrix} \alpha & 2\beta & \gamma+3\lambda & 2\mu+4\varphi & \psi+3\chi & 2\chi' & \varphi' & 0 \\ \beta & 2(\gamma+\lambda) & \delta+4\mu+3\varphi & 2\nu+2\psi+6\chi & \nu'+4\chi'+3\psi' & 2(\varphi'+\mu') & \lambda' & 0 \\ \lambda & 2(\varphi+\mu) & \nu+4\chi+3\psi & 2\nu'+2\psi'+6\chi' & \delta'+4\mu'+3\varphi' & 2(\gamma'+\lambda') & \beta' & 0 \\ \varphi & 2\chi & \psi'+3\chi' & 2\mu'+4\varphi' & \gamma'+3\lambda' & 2\beta' & \alpha' & 0' \\ 0 & \alpha & 2\beta & \gamma+3\lambda & 2\mu+4\varphi & \psi+3\chi & 2\chi' & \varphi' \\ 0 & \beta & 2(\gamma+\lambda) & \delta+4\mu+3\varphi & 2\nu+2\psi+6\chi & \nu'+4\chi'+3\psi' & 2(\varphi'+\mu') & \lambda' \\ 0 & \lambda & 2(\varphi+\mu) & \nu+4\chi+3\psi & 2\nu'+2\psi'+6\chi' & \delta'+4\mu'+3\varphi' & 2(\gamma'+\lambda') & \beta' \\ 0 & \varphi & 2\chi & \psi'+3\chi' & 2\mu'+4\varphi' & \gamma'+3\lambda' & 2\beta' & \alpha' \end{vmatrix} = 0.$$

So far as we know from the actual work the conditions (16) and (22) are only *necessary* conditions for undulations. Hence the question arises, do we obtain by this method a condition which is *sufficient* for an undulation? This question can be answered in the affirmative because in this way an invariant of order $4(n-3)$ of the R^n is obtained which is not an identity and which vanishes when the R^n possesses this singularity. Since it is known* that an undulation of the R^n is conditioned by the vanishing of an invariant of order $4(n-3)$ it is evident that the above method yields an invariant whose vanishing actually conditions an undulation. The same kind of argument may be used to show the sufficiency of the cusp conditions which are to be found in the following pages, and methods of this sort have been used by all writers† on invariants.

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* See Brill, Mathematische Annalen, vol. 12, pp. 107–112; also, Meyer, loc. cit., p. 11.

† Salmon, Higher Algebra, Fourth Edition, p. 190–191; also, W. Stahl, loc. cit., p. 11.

It is necessary in this place to state several important theorems* regarding rational curves and to explain exactly what they mean when applied to the R_2^5. Every rational curve of order n in the plane is symmetrically represented by $n - 2$ binary forms of degree n; all line sections of the R_2^n are sets of points whose parameters are apolar to each of these $(n - 2)$ binary n-ics and therefore apolar to any linear combination of them. The combinants of these $(n - 2)$ binary n-ics are invariants of the R^n and are expressible in terms of the three-rowed determinants of the matrix of coefficients of the parametric equations of the curve by means of a scheme which we shall soon illustrate; also, the combinants of the three binary n-ics in the parametric equations of the curve are invariants of the R^n. When $n = 5$ or if we are considering the R_2^5 a very special relation exists: the R_2^5 is symmetrically represented by three binary quintics other than those which occur in the parametric equations of the R_2^5; the combinants of the three binary quintics in the parametric equations of the R_2^5 are invariants of the R_2^5; also, the general theory states that the combinants of the three binary quintics which are apolar to all line sections of the R_2^5 are invariants of the R_2^5. Let the three† binary quintics which are apolar to all line sections of the R_2^5 be

$$\begin{aligned}
&\alpha_0 t^5 + \alpha_1 t^4 + \alpha_2 t^3 + \alpha_3 t^2 + \alpha_4 t + \alpha_5 = 0,\\
(24)\qquad &\beta_0 t^5 + \beta_1 t^4 + \beta_2 t^3 + \beta_3 t^2 + \beta_4 t + \beta_5 = 0,\\
&\gamma_0 t^5 + \gamma_1 t^4 + \gamma_2 t^3 + \gamma_3 t^2 + \gamma_4 t + \gamma_5 = 0.
\end{aligned}$$

If the combinants of any three binary quintics are given the combinants of the three binary quintics (24) *in terms of the coefficients of* (17) *can be obtained by an easy substitution*‡ which we shall now give. The combinants of (24) are expressible in terms of the three-rowed determinants of the matrix

$$(25)\qquad \begin{Vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 \end{Vmatrix}.$$

The matrix of coefficients of (17) is

$$(26)\qquad \begin{Vmatrix} a_0 & b_0 & c_0 & d_0 & e_0 & f_0 \\ a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \end{Vmatrix}.$$

* Meyer's Apolarität und rationale Curven contains all these theorems; the expressions "symmetrically represented" and "fundamental involution" are used in this paper as he uses them.

† In (24) the binary quintics are purposely written without binomial coefficients.

‡ Meyer, Apolarität und rationale Curven, pp. 1–10.

Suppose the combinants of (24) are given in terms of the three-rowed determinants of the matrix (25); to obtain the combinants of the three binary quintics which symmetrically represent the R_2^5 *in terms of the three-rowed determinants of* (26), it is only necessary to make the substitution of complimentary determinants, i. e., for $|\alpha_0\,\beta_1\,\gamma_2|$ substitute $|def|$, for $|\alpha_2\,\beta_3\,\gamma_5|$ substitute $|abe|$, etc., after which allowance can be made for binomial coefficients, the exact substitution being illustrated in what follows. Since this method holds for any three binary quintics it must hold in particular for the binary quintics in (17). If an invariant of the R_2^5 is given in terms of the determinants of the matrix (26) *it may be looked upon as a combinant of the three binary forms* (17); by substituting $|def|$ for $|abc|$, $|bce|$ for $|adf|$, etc., we obtain the same combinant of the three binary quintics which symmetrically represent the R_5^2 *written without binomial coefficients*; hence by allowing for these coefficients the same combinants of the three binary quintics which symmetrically represent the R_2^5 of (17) are obtained. The exact substitutions which must be made in a combinant of the three binary quintics which occur in the parametric equations of the R_2^5 of (17) in order to obtain the same combinant of the three binary quintics which symmetrically represent the R_2^5 of (17) are in terms of the Greek letters of (20)

$$(27)\quad \begin{array}{lllll} \alpha = 50\alpha', & \beta = 50\beta', & \gamma = 100\lambda', & \delta = 500\varphi, & \lambda = 25\gamma', \\ \alpha' = 50\alpha, & \beta' = 50\beta, & \gamma' = 100\lambda, & \delta' = 500\varphi', & \lambda' = 25\gamma, \\ \mu = 50\mu', & \nu = 250\chi', & \psi = 50\psi', & \varphi = 5\delta', & \chi = 10\nu', \\ \mu' = 50\mu, & \nu' = 250\chi, & \psi' = 50\psi, & \varphi' = 5\delta, & \chi' = 10\nu. \end{array}$$

It might be supposed that each invariant of the R_2^5 by means of the substitutions (27) transforms into a multiple of itself, but this is not always the case. Consider the set of five points cut out of the R_2^5 by the covariant line* of the R_2^5 whose parameters are given by

$$(28)\quad (2\lambda - \gamma)t^5 + (10\varphi - \delta)t^4 + (10\chi - 2\nu)t^3 + (10\chi' - 2\nu')t^2 + (10\varphi' - \delta')t + (2\lambda' - \gamma') = 0.$$

By means of the substitutions (27) the equation (28) transforms into a quintic whose roots are the reciprocals of the roots of (28); hence the invariants of (28) are the same as the invariants of the transformed equation. The same is true of the 9-ic

* The equation of this line is $|afx| - 5|bex| + 10cdx = 0$ and is obtained from the apolarity condition of two line sections by making x_0, x_1, x_2 the coordinates of the point in which the two secant lines intersect.

$$(\alpha t)^3(\beta t)^3(\gamma t)^3 \begin{vmatrix} \alpha_1^2 & \alpha_1\alpha_2 & \alpha_2^2 \\ \beta_1^2 & \beta_1\beta_2 & \beta_2^2 \\ \gamma_1^2 & \gamma_1\gamma_2 & \gamma_2^2 \end{vmatrix} = 0 \tag{29}$$

which yields the 9 flex parameters* of the R_2^5 of (17). But if an invariant of (28) or (29) breaks up into two factors of the same degree it is by no means true that each factor transforms into itself, for each might transform into the other. In fact, this is exactly what does happen for the invariant (22). Each invariant of (29) as a whole must transform into itself by means of (27); the discriminant of (29) breaks up into two factors because it is known that two flexes of a rational curve unite in one way to form an undulation and in another to form a cusp; in the case of the R_2^5 these invariants are of the same degree and it is easy to verify that (22) does not transform into itself by means of (27); hence the transform of (22) by (27) yields the cusp invariant of the R_2^5 of (17). That this fact is a special case of a much more general theorem in regard to plane rational curves will appear when we consider the general cusp condition for rational curves. The cusp condition of the R_2^5 is found by making the substitutions (27) in the expression (22).

The Undulation Invariant of the R_2^6.

If we make $n = 6$ in (1), (2) and (5) and proceed as before the undulation condition of the R_2^6 is found by requiring four 9-ics to be satisfied by the same value of t.

The four 9-ics in expanded form are

$$\begin{aligned} &|abc|t^9 + 3|abd|t^8 + [3|abe| + 6|acd|]t^7 + [|abf| + 8|ace| + 10|bcd|]t^6 \\ &\quad + [3|acf| + 6|ade| + 15|bce|]t^5 + [3|adf| + 6|bcf| + 15|bde|]t^4 \\ &\quad + [|aef| + 8|bdf| + 10|cde|]t^3 + [3|bef| + 6|cdf|]t^2 + 3|cef|t \\ &\quad + |def| = 0 \end{aligned} \tag{34}$$

and

$$\begin{aligned} &|abd|t^9 + 3[abf + |acd|]t^8 + [3|abf| + 6|bcd| + 9|ace|]t^7 \\ &\quad + [|abg| + 8|ade| + 9|acf| + 18|bce|]t^6 \\ &\quad + [3|acg| + 9|adf| + 18|bcf| + 18|bde|]t^5 \\ &\quad + [3|adg| + 3|aef| + 6|bcg| + 15|cde| + 24|bdf|]t^4 \\ &\quad + [|abg| + 7|bdg| + |aeg| + 9|bef| + 8|cdf|]t^3 \\ &\quad + [3|bcg| + 6|cdg| + 9|cef|]t^2 + 3[ceg + |def|]t + |deg| = 0; \end{aligned} \tag{35}$$

* The expanded form of (29) is exactly the same as equation (34) on page 151.

the third 9-ic is the symmetrically formed 9-ic obtained from (35), its first coefficient being $|acd|$ and its last $|dfg|$; the fourth can be obtained as a 9-ic symmetrical with (34). By multiplying these four 9-ics by t^2 and t we obtain 8 equations which taken with the original 4 may be considered 12 11-ics whose determinant equated to zero is the undulation condition of the R_2^6. It is to be observed that (34) is the equation of the 9 flexes of the R_2^5 of (17); the corresponding expression for the R_2^7 is the flex 12-ic of the R_2^6, etc. Hence the problem of finding the $4(n-3)$-rowed determinant whose vanishing is the condition for the R_2^n to have an undulation is reduced to finding one new $(3n-9)$-ic—the correspondent of (35)—provided that the flex equation of the R_2^{n-1} is known.

The Cusp Invariant of R_2^n.

We recall that every R_d^n is symmetrically* represented by $n-d$ binary forms of order n and that combinants of these $n-d$ binary forms are invariants of the R_d^n. Hence in the plane the R_2^n is symmetricaly represented by $n-2$ binary forms of order n; i. e., the R_2^3 is symmetrically represented by a binary cubic whose invariants are invariants of the R_2^3 and since a binary cubic has only one invariant, its discriminant, the condition for the R_2^3 to have a cusp is the vanishing of this invariant; it is known that the condition for the R_2^4 to have a cusp is the condition for there to be a member of the pencil of binary quartics which symmetrically represent the R_2^4 which has a cubed factor; it has just been proved in the previous section of this paper that the condition for the R_2^5 to have a cusp is the condition for there to be a member of the system of binary quintics associated with the R_2^5 which contains a fourfold factor. Hence, the natural thing to suppose is that the R_2^n will have a cusp if only the associated system of $n-2$ binary forms of order n have a member which contains an $(n-1)$-fold factor. Is this true, or not?

That it is only one condition for there to be a member of a system of $n-2$ binary forms which contains an $(n-1)$-fold factor has been sufficiently illustrated in the previous paragraphs. Suppose this condition is imposed and that there is a member of this system of $n-2$ binary n-ics which contains an $(n-1)$-fold factor which may be taken ∞ as well as anything else. This member may be written in the form

$$\alpha_{n-1}t + \alpha_n = 0; \tag{36}$$

this n-ic (36) must be apolar to all line sections of the R_2^n of (1) which requires

$$\alpha_n a_i - \alpha_{n-1} b_i = 0; \qquad i = 0, 1, 2. \tag{37}$$

* Meyer, Apolarität und rationale Curven, p. 9.

Therefore

$$\frac{a_0}{b_0} = \frac{a_1}{b_1} = \frac{a_2}{b_2}. \tag{38}$$

What effect does this have upon the parametric equations of the R_2^n? By reason of (38) these may be written in the form

$$\begin{aligned} x_0 &= a_0t^n + nb_0t^{n-1} + \frac{n(n-1)}{2}c_0t^{n-2} \cdots, \\ x_1 &= ka_0t^n + knb_0t^{n-1} + \frac{n(n-1)}{2}c_1t^{n-2} \cdots, \\ x_2 &= qa_0t^n + qnb_0t^{n-1} + \frac{n(n-1)}{2}c_2t^{n-2} \cdots. \end{aligned} \tag{39}$$

By choosing another triangle of reference $x_0x_1'x_2'$, where $x_2' = x_2 - qx_0$ and $x_1' = x_1 - kx_0$, we may write (39) in the form

$$\begin{aligned} x_0 &= a_0t^n + nb_0t^{n-1} + c_0't^{n-2} \cdots, \\ x_1' &= c_1't^{n-2} \cdots, \\ x_2' &= c_2't^{n-2} \cdots. \end{aligned} \tag{40}$$

Again taking as a new triangle of reference $x_0x_1'x_2''$ where $x_2'' = c_2'x_1' - c_1'x_2'$ we may write the R_2^n

$$\begin{aligned} x_0 &= a_0t^n + nb_0t^{n-1} + c_0't^{n-2} + d_0't^{n-3} \cdots, \\ x_1' &= c_1't^{n-2} + d_1't^{n-3} \cdots, \\ x_2'' &= d_2't^{n-3} \cdots. \end{aligned} \tag{41}$$

which shows that the R_2^n has a cusp, the cusp tangent being $x_2'' = 0$.

Hence, the condition for the R_2^n to have a cusp is the condition for there to be a set of its fundamental involution which contains an $(n-1)$-fold factor.*

The same method that has been used to write out the cusp invariants of the R_2^3, R_2^4, and R_2^5 can be applied generally. Let the $n-2$ binary n-ics whose linear system constitutes the fundamental involution of the R_2^n be written symbolically.

$$\left.\begin{aligned} (\alpha't)^n &= 0, \\ (\beta't)^n &= 0, \\ (\gamma't)^n &= 0, \\ &\cdots\cdots \\ &\cdots\cdots \\ &\cdots\cdots \\ (\pi't)^n &= 0 \end{aligned}\right\} \quad (n-2) \text{ equations.} \tag{42}$$

* The sufficiency of this condition is proved by a course of reasoning similar to that on page 8.

The condition for there to be a member of the linear system of (42) which contains an $(n-1)$-fold factor is the vanishing of the matrix

$$(43)\qquad (\alpha' t)^2(\beta' t)^2(\gamma' t)^2 \cdots (\pi' t)^2 \begin{vmatrix} \alpha_1'^{\,n-2} & \alpha_1'^{\,n-3}\alpha_2' & \cdots & \alpha_2'^{\,n-2} \\ \beta_1'^{\,n-2} & \beta_1'^{\,n-3}\beta_2' & \cdots & \beta_2'^{\,n-2} \\ \gamma_1'^{\,n-2} & \gamma_1'^{\,n-3}\gamma_2' & \cdots & \gamma_2'^{\,n-2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \pi_1'^{\,n-2} & \pi_1'^{\,n-3}\pi_2' & \cdots & \pi_2'^{\,n-2} \end{vmatrix}.$$

The matrix (43) yields $(n-1)$ equations of degree $(2n-4)$ which such a value of t must satisfy; the coefficients of these $(2n-4)$-ics are expressible in terms of the $(n-2)$-rowed determinants of the matrix of coefficients of (42): each of these $(n-1)$ equations has $(2n-3)$ coefficients; if these $(n-1)$ equations are multiplied by t we obtain $(n-1)$ other equations which taken with the original $(n-1)$ may be looked upon as $(2n-2)$ binary $(2n-3)$-ics; the determinant of these $(2n-2)$ equations is of order $2(n-1)$ in the $(n-2)$-rowed determinants of the matrix of coefficients of (42). The $(n-2)$-rowed determinants of this determinant are now replaced by three-rowed determinants of the matrix of coefficients of (1) when $n = n$ and $d = 2$ in this way; suppose a given $(n-2)$-rowed determinant is formed from the one matrix by striking out a certain set of three columns, the determinant to be substituted for it is formed from the other matrix—the matrix of coefficients of (1) for $n = n$ and $d = 2$—by using the constituents of these three columns. In this way it is possible to obtain the cusp condition of an $R_2{}^n$ as the vanishing of a determinant of order $2(n-1)$ whose constituents are the three-rowed determinants of the type $|abc|$. If we understand that in both matrices binomial coefficients have been ignored the invariant obtained in the above maner would be that invariant for the $R_2{}^n$ whose parametric equations do not contain binomial coefficients; hence allowance must be made for these in every case if we wish to find the cusp invariant for the $R_2{}^n$ as written in (1). In the case of the $R_2{}^6$ suppose the four sextics of (42) are

$$(44)\qquad (\alpha' t)^6 = 0, \quad (\beta' t)^6 = 0, \quad (\gamma' t)^6 = 0, \quad (\delta' t)^6 = 0.$$

The cusp condition is imposed if there is a value of t which satisfies the matrix

$$(45)\qquad (\alpha't)^2(\beta't)^2(\gamma't)^2(\delta't)^2 \begin{vmatrix} \alpha_1'^4 & \alpha_1'^3\alpha_2' & \alpha_1'^2\alpha_2'^2 & \alpha_1'\alpha_2'^3 & \alpha_2'^4 \\ \beta_1'^4 & \beta_1'^3\beta_2' & \beta_1'^2\beta_2'^2 & \beta_1'\beta_2'^3 & \beta_2'^4 \\ \gamma_1'^4 & \gamma_1'^3\gamma_2' & \gamma_1'^2\gamma_2'^2 & \gamma_1'\gamma_2'^3 & \gamma_2'^4 \\ \delta_1'^4 & \delta_1'^3\delta_2' & \delta_1'^2\delta_2'^2 & \delta_1'\delta_2'^3 & \delta_2'^4 \end{vmatrix} = 0.$$

If (45) is expanded the result is 5 octavics whose coefficients are four-rowed determinants of the matrix of coefficients of (44); if the coefficients of these octavics are replaced by the three-rowed determinants of the matrix of coefficients of the parametric equations of the R_2^6 in the manner just explained, we have 5 octavics which multiplied by t furnish 5 other equations; these ten equations may be considered 10 9-ics, the determinant of whose coefficients equated to zero is the cusp condition of the R_2^6.

Higher Dimensions.

The methods used in the preceding paragraphs for rational curves in the plane can be applied without change of argument to rational curves in higher dimensions. For instance, in ordinary space an R_3^n has $4(n-3)$ *tetratactic* planes, or planes cutting the curve in four consecutive parameters; hence, by applying the method used to find the undulation condition in the plane we find the invariant which vanishes when the R_3^n has a pentatactic plane, and it can be expressed as a determinant of order $5(n-4)$ whose constituents are the four-rowed determinants of the matrix of coefficients of the parametric equations of the curve; also the singularity in higher spaces which corresponds to the cusp in the plane can be found in an analogous way. Also, as every invariant of the R_2^5 by means of (27) transforms into itself or into another invariant of the R_2^5, so any invariant of an R_k^{2k+1} transforms into itself or into another invariant of the R_k^{2k+1} by means of a scheme analogous to (27).

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